High-order multipole radiation from quantum Hall states in Dirac materials

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I. INTRODUCTION

The advent of graphene and other two-dimensional (2D) materials has significantly increased the number of optically accessible, two-dimensional electron systems that exhibit the quantum Hall effect [1–4]. These materials can be engineered into devices with nearly atomic scale precision, enabling advances in the manipulation and spectroscopy of quantum Hall states [5,6]. As compared to low-frequency transport and electrical control, optical methods do not require ohmic or superconducting contacts and can be reconfigured on submicron-length and sub-ps time scales. Motivated by the prospect for quantum optical manipulation of quantum Hall states in these materials, we investigate fundamental effects in their optical response when the wavelength of light is much less than the size of the sample. This knowledge can be used to design optical-based protocols for spatially resolved manipulation and spectroscopy of quantum Hall states.

Optical studies of quantum Hall systems display a rich phenomenology due to the strong effect the magnetic field has upon the electronic orbitals and levels. For laboratory magnetic fields, intraband Landau-level transitions typically lie in the far-infrared (IR) portion of the electromagnetic spectrum [7–11]. The long wavelength of these transitions enables several novel applications to quantum optics [12–22], but increases experimental difficulty. Interband transitions can cover a wide range of wavelengths depending on the band structure and have been extensively studied in AlGaAs heterostructures for spectroscopy of fractional quantum Hall states [23–29]. Inter-Landau-level transitions in graphene have been spectroscopically probed from terahertz up to optical frequencies [30–38]. In the transition-metal dichalcogenides, the magneto-optical response is typically dominated by excitonic effects due to the large exciton binding energy in these materials [39–45]. However, optical signatures of interband Landau-level transitions have been directly observed in WSe2 [46].

In this paper, we investigate 2D materials whose low-energy band structure can be approximately described by a Dirac model, which we refer to as 2D Dirac materials (2DDMs). We show that the quantum Hall edge states support high-order, radiative multipole transitions. These transitions are a consequence of the large electronic coherence length and topological translation symmetry of the edge states, but have been overlooked in previous treatments of the optical response of quantum Hall systems. Accessing these transitions would allow novel methods for optical spectroscopy and manipulation of integer and, potentially, fractional quantum Hall edge states. On the other hand, the radiation from the bulk of the 2DDMs is dominated by dipole emission, whose spectral properties are correlated with the disorder landscape. We find the conditions under which these bulk optical transitions can be spatially resolved, which enables optical imaging and manipulation of the potential landscape of the quantum Hall states.

Consider a 2DDM in the integer quantum Hall regime with an electron-hole pair excited above the Fermi level. At integer filling, standard arguments show that the majority of the states in the bulk are localized due to disorder [47]. When the localization length of the electron-hole pair is much less than the optical wavelength, the optical radiation in the far field will appear as dipole emission, but with a spectrum that varies with the local disorder potential [see Fig. 1(a)]. This argument demonstrates that spatially mapping out the emission spectrum across the sample will reveal correlations in the disorder on the scale of the optical wavelength.

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As the electron-hole pair approaches the edge, the situation changes dramatically because these states are not localized and exhibit electronic coherence that extends across the entire sample [47]. Furthermore, due to the magnetic field, the edge states carry a large angular momentum. In principle, this angular momentum can be transferred into the optical radiation during emission. Such a transfer process is necessarily associated with the presence of higher-order multipole moments in the far-field radiation.

To examine the nature of the spontaneously emitted radiation, we also decompose the optical field into eigenmodes of $L_z$ about the center of the 2DDM sample with orbital angular momentum (OAM) $\hbar \ell$ and longitudinal momentum $\hbar k$. Such states are known as cylindrical vector harmonics and are closely related to the cylindrically symmetric Laguerre-Gaussian modes within the paraxial approximation [48]. Due to disorder, the electrons on the edge will not be in a pure angular momentum eigenstate, but will be in a superposition of angular momentum states narrowly peaked around the value $m_e \sim r^2/\ell_z^2$, where $r_e$ is the approximate radius of the edge and $m_e$ is the angular momentum quantum number defined in a gauge-invariant manner in Appendix A. The multipole transitions arise because any electron in the conduction band in the angular momentum state $m$ can conserve total angular momentum by recombining with a hole in the valence band in the state $m'$ and emitting light with OAM $\ell = m - m'$ [see Fig. 1(b)]. We find that these transitions are allowed with a nearly uniform branching ratio up to a cutoff give by $2\pi r_e/\lambda$, where $\lambda$ is the optical wavelength. When the dephasing of the electron transport on the edge is included, this scaling should be modified to $\ell_e/\lambda$, where $\ell_e$ is the coherence length of the edge states.

These arguments are quite general and demonstrate that the multipole radiation is a direct consequence of the large electronic coherence length of the edge states. To understand the behavior and scaling of these transitions in more detail, we consider a cylindrically symmetric edge below such that the multipole radiation pattern can be calculated analytically.

II. DIRAC MODEL

We consider the low-energy Dirac Hamiltonian of the form (neglecting spin)

$$H = \hbar \omega_B \mathbf{k} \cdot \mathbf{m} + m_0 v^2 \tau_z,$$  \hspace{1cm} \hspace{1cm} (1)$$

where $v$ is the Dirac velocity, $\mathbf{k} = (k_x, k_y)$ is the in-plane wave vector, $\tau_z = (\tau_x, \tau_y, \tau_z)$ are Pauli matrices operating on the Dirac pseudospin, and $m_0$ is the effective Dirac mass. At zero magnetic field, the spectrum of $H$ is $\epsilon(k) = \pm \sqrt{m_0^2 v^2 + \hbar^2 \omega_B^2 |n|}$, where $n$ is an integer, $\omega_B = \sqrt{2eB/\hbar}$ is the cyclotron frequency, and $\ell_c = \sqrt{\hbar/eB}$ is the magnetic length. Throughout this work, we restrict our discussion to a single valley for simplicity.

The light-matter interaction for $H$ can be found through the usual prescription $\mathbf{k} \rightarrow \mathbf{k} - e\mathbf{A}/c$,

$$H_{\text{int}} = \frac{e^2}{\sqrt{2c}} \left[ \tau_z A_n^+(x, y) + \tau_z A_n^-(x, y) \right] e^{-i\omega t} + \text{H.c.},$$

(2)

where $A_n = (A_n \pm iA_n)/\sqrt{2}$ are the $\sigma_z$ circularly polarized components of the vector potential $A$ in the plane of the 2D material. Due to the Dirac band structure, the pseudospin operators $\tau_z$ couple the $n$th Landau level to both $n \pm 1$ and $-n \pm 1$. This leads to the optical selection rule for $\sigma_\ell$ circularly polarized light: $n \rightarrow n'$ with $|n'| = |n| \pm 1$ [30].

We represent the single-particle states in the symmetric gauge, in which case the eigenstates $|n, m\rangle$ take the form [49]

$$\langle x, y | n, m \rangle \propto \left( \frac{\alpha_n \sqrt{|n|}}{\beta_n \sqrt{2\ell_c}} \right)^{m-n} D_{n-m}^{[-m-n]} \langle u | n+m \rangle e^{-\alpha u^2/4\ell_c^2},$$

(3)

where $u = x + i y$, $D_{n-m}^{[-m-n]}$ acts as a raising operator on the Landau-level eigenfunctions, $(\alpha_n, \beta_n)^T = (0, 1)$, and, for $n > 0$ ($n < 0$), $(\alpha_n, \beta_n)^T$ are the positive (negative) eigenvectors of the $2 \times 2$ matrix,

$$H_n = \begin{pmatrix} m_0 v^2 & \hbar \omega_c \sqrt{|n|} \\ \hbar \omega_c \sqrt{|n|} & -m_0 v^2 \end{pmatrix},$$

(4)

whose eigenvalues are the energy eigenvalues $\epsilon_n$. We represent the OAM eigenstates for the optical field in the basis of cylindrical vector harmonics [48], which take the form $\mathbf{E}(x, y, z) = \sum_{\ell, k} E_{\ell, k}(r) e^{i(k \cdot \mathbf{r} + \ell \theta + ikz)}$, where $r = |\mathbf{r}|$ and $\theta = \tan^{-1}(y/x)$.

III. RADIATION FROM THE EDGE

We first consider the light emission from the edge states of the quantum Hall system. The edge can either be formed by an external confining potential, at an interface with vacuum or another material, or from an abrupt change in the local dielectric environment. An externally applied potential will generally lead to identical confining potentials for the Landau levels in the conduction and valence band. As a result, the optical transitions between edge states will be degenerate with the transitions in the bulk.

In order to selectively address the edge states, it is desirable to have a difference in dispersion between the edge states in the conduction and valence bands [see Fig. 1(b)]. Such a difference in slope can arise at a sharp interface due to local modifications of the band structure [50]. In the case of graphene with a
The first term \( u_0 \) corresponds to long-range diagonal disorder arising from, e.g., charged impurities, while the other terms are associated with shorter-range effects such as, e.g., variations in the two sublattice potentials \( (u_1) \), tunneling rates \( (u_{x,y}) \), or the presence of vacancies and defects.

The projection of \( H_{\text{dis}} \) into the Landau levels leads to smoothing of the disorder on the scale of \( \ell_c \). This produces a potential landscape for each Landau level \( U_n(x,y) = \langle x,y | T_n | P_n H_{\text{dis}}, P_n \rangle | x,y \rangle \), where \( P_n \) is a projector into the \( n \)th Landau level and \( T_n \) traces over the pseudospin states. This landscape gives rise to (1) an adiabatic shift of the edge position and (2) localized states in the bulk. Thus, the edge multipole effects remain the same, while the bulk radiation becomes dominated by transitions between localized states, each with a different spectral signature [see Fig. 1(a)].

To see how these spectral signatures can be used to image the disorder landscape, we consider near-resonant excitation between Landau levels with \( \sigma_+ \) polarized light and a probe whose frequency \( \omega_p \) is scanned through the resonance \( \hbar \omega_0 = \epsilon_{n+1} - \epsilon_{n} \). The disorder in the optical transition frequency, \( U(x,y) = U_{n+1}(x,y) - U_n(x,y) \), for \( n = 0 \) is shown in Fig. 3(a). To obtain the spatial profile of emitted light, we approximate the far-field emission pattern by a convolution of \( U(x,y) \) with the filter function \( n_e(r) = \sin(4\pi r/\lambda)/\pi r^2 \), which arises from the diffraction limit. Here, \( \lambda = (\hbar n_0/c)(\epsilon_{n+1} - \epsilon_{n})^{-1} \) is the central wavelength of emitted light and \( n_0 \) is the index of refraction of the surrounding substrate. We construct the disorder potential by finding the probe frequency at which the local scattered phonon scattering. For integer quantum Hall states in GaAs, the coherence length was measured via transport methods to be at least 10–20 \( \mu \text{m} \) [52], which is much greater than the relevant optical wavelengths.

To understand this effect more quantitatively, we decompose the radiative emission rate \( \gamma_m \) of an excited electron in the state \( |n+1,m\rangle \) into all the multipole moments \( \gamma_m = \sum_{\ell \geq 0} \gamma_{\ell m} \) [53]. Each individual component can be found using Fermi’s golden rule for the emission into the free-space modes with a specified \( \ell \). We give the matrix elements in Appendix B. Two illustrative examples are shown in Fig. 2(c) for the \( n = 0 \) to \( n = 1 \) transition with Dirac parameters for single-layer WSe\(_2\). We plot the branching ratio \( \gamma_{\ell m}/\gamma_m \) for two different edge radii, which confirms the scaling analysis from above. For \( r_e = 1.5 \mu \text{m} \), we find a nearly uniform distribution for the spontaneous emission out to \( \ell = 50 \). Inclusion of disorder will modify the shape of the distributions in Fig. 2(c), but it will not reduce \( \ell_{\text{max}} \), which is simply a result of the large coherence length of the edge states compared to \( \lambda \).

IV. RADIATION FROM THE BULK

We now consider the optical emission from the localized states in the bulk of the 2D material at integer filling. In particular, we show that the disorder landscape can be reconstructed through optical imaging of the scattered light. We can include disorder in the Dirac model by adding all terms consistent with the symmetries of the hexagonal lattice (neglecting intervalley scattering) [54],

\[
H_{\text{dis}} = u_0(r) I + u(r) \cdot \tau.
\]
perturbation theory, we can see from Eq. (4) that for massless Dirac fermions, \( U \) for sufficiently massive Dirac fermions, \( U(r) \) is dominated by \( \tau_z \) disorder, while for for sufficiently massive Dirac fermions, \( U(r) \) is dominated by \( \tau_z \) disorder. A related measurement in massive 2DDMs could be used to indirectly map out the diagonal disorder term \( u_0(r) \) by going away from integer filling. In particular, the exciton binding energy will vary with the local carrier density due to screening effects. Thus, mapping out the exciton line across the sample would reveal spatial variations in the local carrier density, which, in the partially filled, disordered quantum Hall regime, are directly correlated with the underlying disorder potential [58,59].

As we are treating the disorder in degenerate, first-order perturbation theory, we can see from Eq. (4) that for massless Dirac fermions, \( U(r) \) is dominated by the \( \tau_z \) disorder, while for sufficiently massive Dirac fermions, \( U(r) \) is dominated by \( \tau_z \) disorder. A related measurement in massive 2DDMs could be used to indirectly map out the diagonal disorder term \( u_0(r) \) by going away from integer filling. In particular, the exciton binding energy will vary with the local carrier density due to screening effects. Thus, mapping out the exciton line across the sample would reveal variations in the local carrier density, which, in the partially filled, disordered quantum Hall regime, are directly correlated with the underlying disorder potential [58,59].

V. ELECTRON-ELECTRON INTERACTIONS

In our analysis, we have largely neglected the effect of electron-electron interactions on both the disorder landscape and the optically excited electron-hole pair. Near integer filling, the interactions will have a minimal effect on the bare disorder potential because the electronic state is incompressible and cannot screen the disorder [58,59].

The dominant effect of the electron-hole interactions is to lead to Landau-level mixing and magnetoexciton formation, which have to be considered separately for the bulk and the edge. On the edge, magnetoexciton effects are weak because of the predominantly linear dispersion of the edge states. Landau-level mixing can then also be ignored because the electron and hole are both delocalized and interact weakly. For the bulk, our analysis assumes that the magnetoexciton binding energy \( \epsilon_b \) is much less than the strength of the disorder potential. However, in the opposite limit of strongly bound excitons, the \( \tau \) disorder will lead to spatial variations in \( \epsilon_b \). As a result, we expect our conclusions about mapping the \( \tau \) disorder to remain valid in this limit, provided that the disorder potential contains long-range correlations compared to the magnetoexciton Bohr radius.

VI. CONCLUSION

We have studied the properties of the optical radiation from integer quantum Hall edge states in Dirac materials. We showed that the optical emission from the bulk of the 2DDM reflects the disorder landscape and, at the edge, high-order multipole transitions become allowed. As a result, this work establishes that high-order multipole radiation is an important component of the optical spectroscopy and control of quantum Hall states and related topological systems. Furthermore, these large multipole moments may be useful for applications that make use of light with large orbital angular momentum [60]. Although in this work we have focused on effects which are independent of electron-electron interactions, extending the optical spectroscopy and control techniques described here to study fractional quantum Hall systems or magnetoexcitons is a rich avenue for further investigation.

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APPENDIX A: GAUGE-INDEPENDENT DERIVATION OF OPTICAL SELECTION RULES

The Dirac Hamiltonian in the presence of a constant magnetic field in the \( z \) direction can be diagonalized in a gauge-independent manner by introducing the canonical

FIG. 3. (a) The disorder potential \( U(x, y) \) for the interband transitions between Landau levels. (b) \( U(x, y) \) can be reconstructed by correlating the amplitude of spatially resolved scattered light with the frequency of the incoming probe. We took the 2DDM to be embedded in GaP \( (n_0 = 3.2) \) in a 10 T magnetic field with \( \lambda_0 = 1 \mu m \). The optical imaging is able to resolve spatial features down to the diffraction limit \( \lambda_0/2n_0 \approx 160 \) nm.
momentum operators and guiding center coordinate operators \((\hbar = 1)\),
\[
\pi = \mathbf{k} + \frac{eA_0}{c}, \quad (A1)
\]
\[
R = (X, Y) = (x + \ell_c^2 \pi_x, y - \ell_c^2 \pi_y). \quad (A2)
\]
These operators satisfy canonical commutation relations
\([\pi_x, \pi_y] = i/\ell_c^2\) and \([X, Y] = -i\ell_c^2\), which allows one to define commuting bosonic operators associated with these coordinates,
\[
a = \frac{i\ell_c}{\sqrt{2}} (\pi_x + i\pi_y), \quad (A3)
\]
\[
b = \frac{X - iY}{\sqrt{2}\ell_c}. \quad (A4)
\]
In terms of these operators, the Hamiltonian takes the form
\[
H = i\omega_c (a^\dagger a + a a^\dagger) + m_0 \mathbf{v}^2 \tau_z,
\]
which is independent of \(b\). We define the generalized angular momentum operator \([61]\)
\[
L_z = a^\dagger \alpha \mathbf{a} - \beta^\dagger \mathbf{b} - \tau_z/2 + 1/2, \quad (A6)
\]
which commutes with \(H\). In the symmetric gauge, \(L_z = xk_z - yk_z - \tau_z/2 + 1/2\) is equivalent to the usual angular momentum operator with the added term \((1 - \tau_z)/2\). The simultaneous eigenstates of \(H\) and \(L_z\) in the \(K\) valley are defined, for \(n \neq 0\), as
\[
|n, m\rangle = \frac{(\alpha^\dagger)^{|m|-1}(\beta^\dagger)^{m+|n|-1}}{\sqrt{(m+|n|)!\sqrt{|n|!}}} \left( \alpha_n \sqrt{|m+|n|!|n|!} \beta_m^\dagger \alpha^\dagger b^i \right) |0\rangle \quad (A7)
\]
and, for \(n = 0\), as
\[
|0, m\rangle = \frac{(b^\dagger)^m}{\sqrt{m!}} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) |0\rangle. \quad (A8)
\]

To understand the selection rules, we consider a plane-wave incident on the 2DDM with in-plane circular polarization \(\sigma_\perp\) and in-plane wave vector \(k_\perp, \ell\) directed along the \(x\) axis. Using the representation for the position operator \(x = \ell_c (b + b^\dagger + a + a^\dagger)/\sqrt{2}\), we can write the light-matter interaction in a frame rotating with the optical field in terms of the quantum Hall creation and annihilation operators,
\[
H_{\text{int}} = A_0 \tau_+ e^{-ik_\perp \ell \xi(b + b^\dagger + a + a^\dagger)/\sqrt{2}} + \text{H.c.}. \quad (A9)
\]
In this representation, we can see that the plane wave acts as a product of coherent-state displacement operators \(D_\alpha(a)D_\beta(a)\) with amplitude \(\alpha = ik_\perp \ell \xi / \sqrt{2}\), i.e.,
\[
a e^{-iq_\perp \xi (a + a^\dagger)} \sqrt{2} |0\rangle = a D_\alpha(a) |0\rangle = \alpha |\alpha\rangle. \quad (A10)
\]
Focusing on the \(n = 0\) state for simplicity, we see that acting with \(H_{\text{int}}\) on \(|0, m\rangle\) leads to the state
\[
H_{\text{int}}|0, m\rangle = A_0 D_\alpha(a) \left( \begin{array}{c} (b^\dagger)^m \\ 0 \end{array} \right) \left( \begin{array}{c} D_\beta(a) \\ 0 \end{array} \right) |0\rangle = A_0 \left( \begin{array}{c} (b^\dagger - \alpha^\dagger)^m \\ 0 \end{array} \right) \left( \begin{array}{c} D_\alpha(a)D_\beta(a) \\ 0 \end{array} \right) |0\rangle. \quad (A11)
\]
To evaluate the selection rules, we first note that we can neglect the effect of the displacement operator \(D_\alpha(a)\) in the second line of Eq. (A11) because \(|\alpha| < \sqrt{2}\ell_x/\ell \ll 1\) (here the first inequality follows because \(k_\perp < 2\pi/\lambda\)). Surprisingly, however, one is not justified in neglecting \(a\) in either the prefactor of this expression or in \(D_\beta(a)\). To understand this result, we expand Eq. (A11) into the basis \(|1, m\rangle\) as
\[
H_{\text{int}}|0, m\rangle \approx A_0 \sum_{j=0}^m \left( \begin{array}{c} (b^\dagger)^{m-j} (-\alpha^\dagger)^j \end{array} \right) \frac{1}{\sqrt{m!}} |0, \alpha\rangle = A_0 \alpha j e^{-|\alpha|^2/2} \sum_{\ell=0}^m F_{\ell, \alpha}(|\alpha|)|1, m + \ell\rangle, \quad (A12)
\]
\[
F_{\ell, \alpha}(\alpha) = \sqrt{(m+\ell)!/m!} \alpha^\ell \sum_{j=0}^m \left( \begin{array}{c} m+j \end{array} \right) (-1)^{|\alpha|^2} (\ell + j)! \right). \quad (A13)
\]
where \(j = \max(0, -\ell)\). Evaluating this sum and using Sterling’s formula \(n! \approx \sqrt{2\pi n} (n/e)^n\), we find that the multipole moments are actually perturbative in \(r_\alpha k_\perp / \ell = \sqrt{m} \ell \xi k_\perp / \ell\) and not \(\ell \ksi / \ell\), as one would naively expect. In particular, in the regime where \(r_\alpha k_\perp / \ell < 1\), we find the scaling
\[
\langle 1, m + \ell | H_{\text{int}} | 0, m \rangle \sim \frac{r_\alpha k_\perp}{\ell}, \quad (A14)
\]
which is identical to the scaling we find for the cylindrical vector harmonics in this regime.

For \(r_\alpha k_\perp / \ell > 1\), one has to use the nonperturbative expression from Eq. (A13) to evaluate the multipole transition moments. Similar to the multipole radiation that we found for the cylindrical vector harmonics, one finds (after averaging over \(k_\perp\)) that this expression is approximately independent of \(\ell\) in this regime. Thus we see that the gauge-independent representation of the plane-wave response is nearly identical to the response we found for the cylindrical vector harmonics discussed in the main text.

**APPENDIX B: SPONTANEOUS EMISSION OF EDGE STATE IN SYMMETRIC GAUGE**

In this section, we define the cylindrical vector harmonic solutions to Maxwell’s equations. We quantize these modes, give the expressions for the matrix elements used to calculate the spontaneous emission of the edge states, and evaluate the scaling of the spontaneous-emission rate with increasing OAM.

To construct the cylindrical vector harmonics, we start with the cylindrically symmetric solutions to the Helmholtz equation,
\[
(V^2 + k_z^2) \psi_{\ell, k}(r) = 0, \quad (B1)
\]
which take the form
\[
\psi_{\ell, k}(r, \theta, z) = e^{ik_z z + i\ell \theta} J_\ell(k_\perp r). \quad (B2)
\]
Here, \((r, \theta, z)\) are the cylindrical coordinates such that \((x, y, z) = (r \cos \theta, r \sin \theta, z)\), \(\ell\) is an integer that labels the orbital angular momentum, \(k\) is the longitudinal wave vector, \(k_\perp = \sqrt{k_0^2 - k^2}\), and \(J_\ell(\cdot)\) are the Bessel functions of the first
We can use these solutions to construct a complete basis for the transverse solutions to Maxwell’s equations in free space in terms of the vector potential in the Coulomb gauge,

\[ A_{\ell,k}^1 = A_0 M_{\ell,k}, \]
\[ A_{\ell,k}^2 = A_0 N_{\ell,k}, \]

where \( A_0 \) is the amplitude. The energy density of \( A_{\ell,k}^\ell \) is given by

\[ u = \frac{\omega^2 e_0}{2k_F} (|M_{\ell,k}|^2 + |N_{\ell,k}|^2) |A_0|^2. \]

We quantize these modes by placing them in a large cylindrical box of radius \( R \) and length \( L \). After quantization, we use the condition \( \int d^3 r u = \hbar \omega \), where \( \omega = c k_F \), to obtain

\[ A_0 = \sqrt{\frac{\hbar k F}{2\pi L R \omega}}. \]

The key quantities that enter the calculations of the main text are the dipole matrix elements between the different Landau-level states. We now give explicit expressions for the matrix elements between the different Landau-level states. We now give explicit expressions for the dipole matrix elements between the different Landau-level states. We now give explicit expressions for the dipole matrix elements between the different Landau-level states. We now give explicit expressions for the dipole matrix elements between the different Landau-level states.

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\[ \gamma_\ell \sim \text{constant,} \quad \gamma_\ell \sim (k_0 r_m/\ell)^{2\ell} \]

\[ \delta_\ell = (\hat{k} \pm i\hat{y})/\sqrt{2} = e^{\pm i\theta}(\hat{k} \pm i\hat{y})/\sqrt{2}, \]

where we used Stirling’s approximation from above. In this regime, \( k_0 r_m \gg 1 \).

\[ \gamma_\ell = 2\pi \sum_{k,l,i} |M_{m,m+\ell}^{l,i}|^2 \delta(c\sqrt{k^2 + k_{\perp}^2} - E_1 + E_0). \]

The quantity \( \gamma_\ell / \sum \gamma_\ell \) is plotted in Fig. 3(c) of the main text. To understand the scaling predicted by this equation, we note that in the generic case where \( \ell, \ll \lambda \) and \( \ell \ll m,\), we can approximate the integral in Eq. (B15) by replacing the photonic mode by its value at \( r = r_m \). This follows because the mode function \( A_{\ell,k}^\ell \) varies on the scale of \( 1/k_F \), so it can be pulled out of the integral over the electronic wave functions, which are peaked at \( r = r_m \) with a width given by \( \ell, \). This implies the scaling

\[ |M_{m,m+\ell}^{l,i}|^2 \sim |J_\ell(k_0 r_m)|^2. \]

As a result, we can find the scaling of \( \gamma_\ell \) by looking at the different scalings of the Bessel function. This is illustrated in Fig. 4 in the regime \( k_0 r_m \gg 1 \). For \( k_F r_m \ll \ell^2 \),

\[ |M_{m,m+\ell}^{l,i}|^2 \sim \cos^2(k_\perp r_m - \pi \ell/2 - \pi/4), \]

which oscillates with \( \ell \). However, in evaluating \( \gamma_\ell \), we average over \( k_\perp \), which washes out these oscillations. As a result, in this regime, \( \gamma_\ell \) is approximately independent of \( \ell \), in agreement with the full calculations shown in Fig. 3(c) of the main text. In the opposite limit \( k_F r_m \gg \sqrt{\ell} \),

\[ |M_{m,m+\ell}^{l,i}|^2 \sim (k_0 r_m/\ell)^{2\ell} \sim (k_0 r_m/\ell)^{2\ell}, \]

where we used Stirling’s approximation from above. In this regime, \( \gamma_\ell \) recovers the typical behavior for higher-order multipole transitions and decreases exponentially with \( \ell \).
[53] Here the sum is restricted to positive $\ell$ because the transitions to negative $\ell$ are forbidden from Pauli blocking for the configuration in Fig. 1(c).