Many-body Chern number from statistical correlations of randomized measurements

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One of the main topological invariants that characterizes several topologically-ordered phases is the many-body Chern number (MBCN). Paradigmatic examples include several fractional quantum Hall phases, which are expected to be realized in different atomic and photonic quantum platforms in the near future. Experimental measurement and numerical computation of this invariant is conventionally based on the linear-response techniques which require having access to a family of states, as a function of an external parameter, which is not suitable for many quantum simulators. Here, we propose an ancilla-free experimental scheme for the measurement of this invariant, without requiring any knowledge of the Hamiltonian. Specifically, we use the statistical correlations of randomized measurements to infer the MBCN of a wavefunction. Remarkably, our results apply to disk-like geometries that are more amenable to current quantum simulator architectures.

Introduction.— Topologically ordered systems are a class of gapped quantum phases of matter [1, 2], which can have robust topological ground-state degeneracy, and host excited states with fractional statistics, known as anyons [3]. These systems, unlike symmetry protected topological (SPT) phases that have short range entanglement, acquire long-range entanglement which makes them a suitable platform for realizing quantum computation [4, 5]. Paradigmatic examples of chiral topologically ordered systems are the fractional quantum Hall (FQH) states that in certain cases are characterized by the many-body Chern number (MBCN), as their topological invariant.

In recent years, the interest in engineering topological states of matter in synthetic quantum systems has substantially grown. Examples of such quantum simulators include neutral atoms [6], superconducting qubits [7, 8], photons [9], and more recently Rydberg atoms [10, 11]. With these developments, the benefit of having direct access to the wave function in quantum simulators opens new avenues to investigate and measure the topological properties. In the conventional condensed matter physics, the detection of topological properties relies on the application of external probes and linear response framework, and similar schemes have been also proposed for the simulated matter [12–14]. Moreover, ancilla-based approaches have been proposed that involve a many-body Ramsey interferometry to measure the topological charge [15], and entanglement spectrum [16]. But the fact that the ancilla should be coupled to the entire system limits the applicability of such schemes. Recently, this question has been theoretically investigated in the context of SPT systems [17–22], but the problem for topologically-ordered system has been relatively unexplored.

Here, we propose a novel method for the measurement of MBCN. Using our recent findings [23], we show that given a wave function on a disk-like geometry, for a single set of parameters, one can construct the MBCN by applying certain operators on the wave function, without knowledge of the Hamiltonian. This should be contrasted with the randomized measurement scheme. We define two regions $R_1$ (red) and $R_2$ (green) in the lattice with side length $l_1 \times l_y$ and $l_2 \times l_y$ respectively. We prepare two identical wave functions $|\psi_A\rangle$ and $|\psi_B\rangle$ in experiment $A$ and $B$ respectively. The local unitary operator $\hat{V}$ is applied in the region $R_1$ in the exp. 1. Subsequently, the random unitary $\hat{U}_{R_1}$ is applied in the region $R_1$ on both wave functions. The projective measurements on the particle occupation basis are performed on regions $R_1$ and $R_2$ in both experiments. The MBCN can be inferred from the statistical correlation between the randomized measurement results in experiment $A$ and experiment $B$. 

![FIG. 1: The randomized measurement scheme.](image-url)
with the common situation where one requires a family of many-body wave functions, e.g., different twist angles on a torus. Importantly, such a construction allows one to perform the measurements using random unitaries [24–26]. Our scheme requires only a single wave function at a given time, for the same set of parameters, as schematically shown in Fig. 1. In other words, in each experimental realization, one requires only a single copy of the system, and simultaneous access to several identical copies of the wave function is not required. Therefore, this scheme can be easily implemented with the state of the art ultracold atoms, Rydberg arrays and circuit-QED platforms.

First, in the context of topological quantum field theory (TQFT) [27], we interpret and generalize the polarization formula for the MBCN [23]. We demonstrate that by introducing two symmetry defects, in the space-time manifold, one can evaluate the MBCN, as an expectation value of symmetry defect operators. This allows us to effectively change the boundary conditions of the wave function. Then, by cutting and gluing space-time manifolds, we show that topologically non-trivial space-time manifolds, such as a torus, can be obtained from a given wave function on a rectangular geometry. Such operations can be obtained by applying a SWAP operator between two subregions [21]. Similar to the Renyi entropy, where the expectation of the SWAP operator can be evaluated using a single copy of the wave function at a time, we show how such space-time surgery can be implemented in an experimental setting. Importantly, we show that the symmetry defects can be implemented by post-processing the data.

As a prerequisite for our protocol, we need to know the number of flux quanta that must be adiabatically inserted into a region of the system before a topologically trivial excitation is obtained [23]. As another feature of our protocol, we note that the amplitude of the SWAP expectation value decreases exponentially with the sub-regions area, in the absence of spatial symmetries. Moreover, the number of randomized measurements increases exponentially with the system size. Therefore, for both reasons, our protocol is particularly suitable for Noisy Intermediate-Range-Quantum (NISQ) devices [28].

Many-Body Chern Number.—In order to introduce the MBCN, we first consider a full multiplet of \( s \) topologically degenerate ground states on a torus. The wave functions are \( \Psi_{\alpha}(\phi_x, \phi_y) \) defined on a torus geometry, with length \( L_x \) and \( L_y \) along the \( x \) and \( y \) directions, respectively. Here \( \alpha = 1, \ldots, s \), and we consider abelian quantum Hall states with Hall conductance \( \sigma_{xy} = \frac{e^2}{h} \), where \( p \) and \( q \) are co-prime integers and the parameter \( s = q \). In this case, the parameter \( s \) is the number of flux quanta that has to be inserted before a topologically trivial excitation is obtained. We note that in general, the parameter \( s \) can be different from \( q \) when the degenerate ground state subspace is composed of multiple topological sectors. [45].

The twisted boundary conditions are defined as \( i\tilde{t}_j(L_k k)\Psi(\phi_x, \phi_y) = e^{i\phi_x} \Psi(\phi_x, \phi_y) \), where \( k = x, y \) and \( \tilde{t}_j(\tau) \) being the magnetic translation operator of the \( j \)th particle along the direction \( \tau \). The MBCN of a FQH system is of the form [29]

\[
C = \frac{1}{2\pi i} \int_0^{2\pi} d\phi_x \int_0^{2\pi} d\phi_y \mathcal{F}(\phi_x, \phi_y),
\]

where \( \mathcal{F}(\phi_x, \phi_y) = -\langle \partial_{\phi_y} \Psi_{\alpha} | \partial_{\phi_y} \Psi_{\alpha} \rangle - \langle \partial_{\phi_y} \Psi_{\alpha} | \partial_{\phi_y} \Psi_{\alpha} \rangle \) is the Berry curvature obtained from adiabatically varying the twist angle boundary conditions \( (\phi_x, \phi_y) \), for a single wave function \( |\Psi_{\alpha}\rangle \).

Alternatively, one can obtain the MBCN, when the wave function is given only as a function of one twist angle. Specifically, let \( |\Psi_{\alpha}(\theta_x)\rangle \) be the ground state wave function in the presence of a flux through the \( x \) direction \( \mathcal{A}_x = \theta_x \), and we take the flux in the \( y \) direction to be zero, \( \mathcal{A}_y = 0 \). We note that for the following argument, one can also consider a cylinder instead of a torus. Following Resta [30], we define the polarization operator as \( R_y = \prod_{x,y} e^{i\pi/2 \hat{\theta}_x} \), where the product is taken over the whole system. We then compute

\[
T(\theta_x, s) = \langle \Psi(\theta_x) | R_y^s | \Psi(\theta_x) \rangle.
\]

Adiabatically changing \( \theta_x \) is equivalent to applying an electric field \( E_x \), which induces a current in the \( y \) direction due to the Hall conductivity, which corresponds to a changing polarization along the \( \hat{y} \) direction. The MBCN therefore can be obtained as

\[
C = \frac{1}{2\pi} \frac{d}{d\theta_x} \text{arg} T(\theta_x, s).
\]

We note that equation above converges to the MBCN in the thermodynamic limit. For systems with finite size, a more robust result can be obtained by averaging over the twist angle: \( C = \frac{1}{2\pi} \frac{d}{d\theta_x} \text{arg} T(\theta_x, s) \). The Hall conductivity corresponds to \( \sigma_H = \frac{C}{\pi} \).

We note Eq. (1) and Eq. (2) are equivalent to each other and require toroidal and cylindrical geometries, respectively. While there are theoretical proposals to implement such geometries [31, 32], an experimental realization remains challenging.

TQFT generalization of Resta Formula.—We interpret and generalize the polarization formula (2) using the TQFT formalism and the Chern-Simons response theory. The low-energy response of the system can be encoded in an effective action for the background electromagnetic gauge field \( A \), such that the TQFT partition function on a space-time manifold \( M \) is given by,

\[
\mathcal{Z}(M, A) = \mathcal{Z}(M, 0) e^{i \xi S_{CS}[A]}.
\]

The Chern-Simons response action is given by \( S_{CS}[A] = \frac{\xi}{4\pi} \int_M e^{i\mu A_\mu} \partial_\mu A^\mu \), where \( \mu = t, x, y \). The space-time
Under these conditions, the partition function is given by

\[ Z = \text{symmetry defects} \]

contractible loops on a disk geometry. We start from two topologically non-trivial manifold, by starting with the state on system A.

The background gauge fields in Eq. (4) form two symmetry defects which are wrapped around two distinct periods of the wave function and therefore, the winding number of arg[\( T(\theta_x) \)] corresponds to the MBCN. We note that while our TQFT derivation of this formula is applicable to cylindrical geometries, extensive numerical simulations indicates that the same formula can also be applied to disk-like geometries [23].

Randomized Measurement Scheme.— We now present the experimental protocol to measure the MBCN via random measurements. Eq. (6) involves the SWAP operator between two copies of the wave function, and the expectation value can be obtained by performing a beam-splitter interaction between the two copies and a parity measurement [33–36]. In contrast, we show that a random measurement protocol requires only a single wave function, at a given time. Our key observation is that, without the symmetry defect operators, Eq. (6) is reminiscent of the second Renyi entropy expression and its evaluation through the SWAP operator expectation value, which can be extracted using randomized measurement [24]. Here, we need to generalize that scheme to incorporate the symmetry defect operators.

Let us consider a two-dimensional square lattice system with open boundary condition. Eq. (6) involves non-local SWAP operations between two replica of the wave functions. It can be performed through the following two randomized measurements as described in Fig. 1.

We start by preparing the wave function |\( \psi \rangle \) in the open boundary condition. We first apply the operator \( \hat{V}_R \) on the state in the experiment A. We then perform the random unitary operation \( \hat{U} \) and the measurements on the occupation probability in the region \( R_1 \) and \( R_2 \) for both experiment A and B. The random unitary operations are sampled from an approximate unitary 2-design [37, 38]. After repeating the measurement \( N_M \) times, we obtain the probability distribution over the occupation basis |\( b \rangle \). The results of the two experiments are \( P_U(b) = |\langle b | \hat{U} |\psi \rangle|^2 \) and \( P_R(b) = |\langle b | \hat{R} \hat{U} |\psi \rangle|^2 \) respectively. We repeat the two experiments with different random unitary operations \( \hat{U} \) for \( N_U \) times. The statistical
correlation of the measurement results in the experiment A and B gives
\[
\mathcal{T}(\theta_x) = \sum_{\{b\}} \sum_{\{b'\}} O_{b,b'}(\theta_x) P_{U}(b) P_{U}(b'),
\]
(7)
where the bar, $\bar{\cdot}$, means the average over the random unitaries from an approximate unitary 2-design. The coefficient $O_{b,b'}(\theta_x) = \delta_{N_1(b),N_1(b')} D_b (-D_{b'})^{N_2(b')-N_2(b')} e^{i\{N_2(b)-N_2(b')\} \theta_x}$, where $N_1(b)$ and $N_2(b)$ are the number of particles of the basis state $|b\rangle$ in the region $R_1$ and $R_2$ respectively and $D_b = \left(N_1(b)\right)$. Since $\mathcal{T}(\theta_x) = \mathcal{T}(\theta_x)$ for an ensemble average over a unitary 2-design [46], the winding number of the measurement result $\arg[\mathcal{T}(\theta_x)]$ gives the Chern number $\hat{C}$.

In the following, we consider the randomized measurement scheme for system with non-trivial Chern number with finite number of $N_c$ and number of projective measurements $N_M$ for each realization of randomized measurement.

**Numerical results.** We present the measurement of MBCN for bosonic fractional quantum Hall states with filling $\nu = 1/2$. We consider hard-core boson on the $N_x \times N_y$ square lattice in the open boundary condition, with a magnetic tunneling Hamiltonian of the form
\[
H_t = -J \sum_{x,y} (\hat{a}_{x+1,y} \hat{a}_{x,y} + e^{-i\Phi} \hat{a}_{x,y+1} \hat{a}_{x,y}) + \text{h.c.},
\]
(8)
where $\hat{a}_{x,y}$ ($\hat{a}_{x,y}^\dagger$) is the bosonic annihilation (creation) operator on site $(x,y)$, $\Phi = 2\pi / q$ is the magnetic flux on each plaquette. The ground state is known to be a FCI state by decreasing the strength $|\Delta|$ is comparable to $T^{-1}$ and $J$, the random quench unitary operator gives the approximate 2-design unitary [25].

The performance of the randomized measurement is characterized by the probability of obtaining the correct MBCN $P[C = 1]$. In Fig. 3(b), we consider the limit of $N_M \to \infty$, the performance of the randomized measurement weakly depends on the number of qubits in the measurement region $R_1$ and $R_2$. In Fig. 3(c) and (d), the shot-noise of the measurements are taken into account. When the number of measurements $N_M$ is of the same order of magnitude as $2^{N_q}$, where $N_q = 0.5n_1 + n_2$, and $n_1$ and $n_2$ are the number of sites in the region $R_1$ and $R_2$ respectively, the probability $P[C = 1]$ starts to saturate. The factor $2^{N_{\text{site}}}$ originates from the birthday paradox scaling of the randomized measurement in the region $R_1$ and the factor $2^{N_{\text{site}}}$ is contributed by the shot-noise of the number operator measurement in the region $R_2$. The randomized measurements can be realized in the current and near-term experimental platform. For example, in the circuit QED architecture with 10kHz repetition rate, each randomized measurement can be performed within a few minutes.

**Adiabatic preparation of FCI.** In order to experimentally realize the above FCI state, one can start by a Mott insulator state, by adding a trapping potential in form of a superlattice $V_{\text{trap}} = M \sum_{x,y} (-1)^{x+y} \hat{n}_{x,y}$, where and $p_c = \text{floor}(x/q)$ [39, 40]. Specifically, the Mott insulator for large trapping strength $M$, can be adiabatically melted into a FCI state by decreasing the strength.

![FIG. 3: Simulation results for Eq. (6) and (7), for the FCI phase with $C = 1$. (a) Obtained MBCN by Eq. (6) for various region size ($\ell_1$, $\ell_2$) and $\ell_3$ with $N_y = 6$, $N_x = 8$, labeled with different markers. (b) Probability of obtaining the expected MBCN $P[C = 1]$ from Eq. (7), using randomized measurements, as a function of the number of random unitary operations $N_U$ with $N_M = \infty$. Region sizes are taken to be $\ell_1 = \ell_2 = 2$. (c, d) Probability of obtaining the expected MBCN versus number of measurements $N_M$, for two sets of region sizes. For all panels, $J = 1$, and $\Phi = 2\pi / 3$. The probability $P[C]$ is computed by averaging over 500 times independent randomized measurement results. Random quench parameters are $\eta = 20$, $\Delta = J$, $T = J^{-1}$ and $n' = 0.5n_1 + n_2$.](image-url)
In Fig. 4, we consider an adiabatic process with $M(t) = M_0(1 - t/T_{ad})$, where $M_0$ is the initial trapping strength and $T_{ad}$ is the adiabatic preparation time. The energy gap remains open in the process of adiabatic preparation, as shown in Fig. 4(b). For slow enough sweep, the overlap between the instantaneous ground state and the adiabatic wave function remains higher than 0.999 for $T_{ad} = 100J^{-1}$, as shown in Fig.4 (c). In Fig. 4(d), we show that the randomized measurement results agree with the theoretical values, in both trivial and topological limits. However, if the sweep time is not in the adiabatic limit, e.g., $T_{ad} = 1J^{-1}$ as in Fig. 4 (c) and (d), then the system deviates from the FCI phase.

**FIG. 4:** Adiabatic preparation of FCI and measurement of MBCN by varying a trapping potential. (a) Schematics of the system considered in the simulation, with open boundary condition and $N_x = 6, N_y = 8$. Region lengths are $\ell_x = 2, \ell_y = 2, \ell_y = 6$ and $\Phi = 2\pi/3$. The potential energy is $-M + M$ in the dark/bright regions, respectively. (b) The energy spectrum, and the corresponding MBCN as a function of $M$. (c) The fidelity $1 - |\langle \psi_{in}|\psi_{ad} \rangle|$, where $|\psi_{in} \rangle$ is the instantaneous eigenstate and $|\psi_{ad} \rangle$ is the adiabatic wave function prepared by linearly decreasing the trapping strength $M$, with the preparation time $T_{ad}$. (d) The MBCN randomized measurement results with parameters $N_M = 1024$, $N_M = 2^n$, $\eta = 20$, $\Delta = 1$ and $T = 1$.

**Outlook:** Our work opens up a new avenue for creating non-trivial topology on space-time manifold, using the SWAP operation. It is particularly intriguing that the SWAP operation can be implemented by random unitaries in the NISQ devices. More broadly, quantum simulators are poised to realize topologically-ordered states that might not occur in a conventional electronic matter. Given this opportunity, it is important to develop measurement methods that go beyond linear response formalism. For example, it is interesting to investigate whether the application of SWAP operator through randomized measurement can be used to probe other topological characterizations, such as modular matrices [11], topological entanglement entropy [42, 43] and the order parameter of the symmetry enriched topological phases [44].

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[45] We define a topological sector to consist of all the degenerate ground states which can be related to each other.
under the operation of quantized flux insertion in the $x$ and $y$ cycles of a torus [23]. See supplementary for the derivation.

Supplementary : Derivation of Eq. (7) In this supplementary, we derive Eq.(7) in the main text, which allows us to extract the MBCN from the statistical correlations of randomized measurements.

We start by considering two arbitrary density matrices $\rho_A$ and $\rho_B$ with dimension $\mathcal{D}$. As shown in [24], the expectation value of the SWAP operator can be implemented by the 2-design random unitary operations of the form

$$\text{tr}(\rho_A \otimes \rho_B \hat{S}_{A,B}) = \mathcal{D} \sum_{s_1,s_2} (-\mathcal{D})^{1-\delta_{s_1,s_2}} P^A_U(s_1) P^B_U(s_2),$$

(10)

where $P^\alpha_U(s) = \text{tr}(U \rho_\alpha U^\dagger \ket{s}\bra{s})$ and $\alpha = A, B$.

Notice that the right hand side of the above equation can be written as

$$\mathcal{D} \sum_{s_1,s_2} (-\mathcal{D})^{1-\delta_{s_1,s_2}} P^A_U(s_1) P^B_U(s_2)$$

$$= \mathcal{D} \sum_{s_1,s_2} (-\mathcal{D})^{1-\delta_{s_1,s_2}} \times \text{tr}((s_1\langle s_1| \otimes |s_2\rangle\langle s_2| U \otimes U \rho_A \otimes \rho_B U^\dagger \otimes U^\dagger)$$

$$= \text{tr}(U \otimes U^\dagger \hat{S}_{A,B} \otimes \hat{S}_{A,B}),$$

(11)

where $\hat{O} = \mathcal{D} \sum_{s_1,s_2} (-\mathcal{D})^{1-\delta_{s_1,s_2}} \langle s_1| \otimes |s_2\rangle\langle s_2|$. 

By using the 2-design property of the random unitary operation, we obtain

$$U \otimes U \hat{O} U^\dagger \otimes U^\dagger = \frac{1}{\mathcal{D}^2 - 1} (\text{tr}(\hat{O}) - \frac{1}{\mathcal{D}} \text{tr}(\hat{S}_{A,B} \hat{O} )) \hat{I}$$

$$+ \frac{1}{\mathcal{D}^2 - 1} (\text{tr}(\hat{S}_{A,B} \hat{O} )) \hat{I}$$

(12)

Since $\text{tr}(\hat{O}) = \mathcal{D}$ and $\text{tr}(\hat{O} \hat{S}_{A,B}) = \mathcal{D}^2$, we arrive at equation (10).

Using equation (10), now we can rewrite $\mathcal{T}(\theta_x)$ defined in the main text, in terms of statistical correlations,

$$\mathcal{T}(\theta_x) = \langle \tilde{\psi}_A| \tilde{\psi}_B| \tilde{W}_{R_2^B}(\theta_x) \tilde{S}_{R_2^A,R_1^B} \tilde{W}_{R_2^A}(\theta_x)| \tilde{\psi}_A| \tilde{\psi}_B)$$

(13)

where $|\tilde{\psi}_A\rangle = \tilde{V}_{R_2^A}|\psi\rangle$.

Since the operators $\tilde{W}_{R_2^A}$ and $\tilde{W}_{R_2^B}$ are diagonal in the particle occupation basis, we can apply it after projective measurements. Specifically, we use the following identity

$$\mathcal{T}(\theta_x) = \sum_{s_{R_2}} e^{iN(s_{R_2})\theta_x} e^{-iN(s_{R_2})\theta_x}$$

$$\times \langle \tilde{\psi}_A| \tilde{\psi}_B| \tilde{S}_{R_2^A,R_1^B} \tilde{P}_{s_{R_2}}^A \tilde{P}_{s_{R_2}}^B | \tilde{\psi}_A| \tilde{\psi}_B)$$

(14)

where $s_{R_2}$ is the basis of particle number configuration in the region $R_2$, $N(s)$ is the number of particle in the basis state $|s\rangle$ and $\tilde{P}_{s_{R_2}}^\alpha$ is the projector that projects
the particle configuration in the region $R_2$ into the state $s_{R_2}$.

Now, we perform the partial trace operator on wave functions $A$ and $B$ and only keep the region $R_1$. Correspondingly, we can rewrite (14) as,

$$\mathcal{T}(\theta_x) = \sum_{s_{R_2}} e^{iN(s_{R_2})\theta_x} e^{-iN(s'_{R_2})\theta_x} \times \text{tr}(\rho_A;_{s_{R_2}} \otimes \rho_B;_{s_{R_2}} \hat{S}_{R_1}^{R_1}, R_1^p),$$

(15)

where the reduced density matrix is defined as $\rho_{p;_{s_{R_2}}} = \text{tr}_{R/R_1}(\hat{P}_{s_{R_2}} |\psi_p \rangle \langle \psi_p |)$, where $p = A, B$.

For the ground state of a number conserving Hamiltonian, the reduced density matrix $\rho_{p;_{s_{R_2}}}$ can be written as the direct sum of the density matrix with different number of particles. Therefore, $\rho_{p;_{s_{R_2}}} = \bigoplus_{k=0}^{\infty} \rho^k_{p;_{s_{R_2}}}$, where $\rho^k_{p;_{s_{R_2}}}$ is the reduced density matrix with $k$ particle in the region $R_1$.

The SWAP operation $\hat{S}_{R_1}^{R_1}, R_1^p$ is a number conserving operation and it can also be factorized into the direct sum of SWAP operations with different number of particles sectors $\hat{S}_{R_1} = \bigoplus_{k=1}^{\infty} \hat{S}_{R_1}^k$. Equation (15) can be simplified as

$$\mathcal{T}(\theta_x) = \sum_{s_{R_2}} e^{iN(s_{R_2})\theta_x} e^{-iN(s'_{R_2})\theta_x} \times \bigoplus_{k=0}^{\infty} \text{tr}(\rho^k_A;_{s_{R_2}} \otimes \rho^k_B;_{s_{R_2}} \hat{S}_{R_1}^k).$$

(16)

The SWAP operator in the $k$-particle sector can be implemented by equation (10) with the random unitary acting on the $k$-particle sector $U_k$ which can be implemented with the random quench dynamics with number conserving Hamiltonian. By replacing the SWAP operation in equation (16) by the random unitary operation defined in equation (10), we reach equation (6) in the main text.