I. Estimation precision scaling without mode-entanglement

Here we derive the precision scaling for estimating a linear combination of parameters \( q = \sum_j w_j \theta_j \) without having mode entanglement. We first consider an input product state \( |\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_d\rangle \), with a nonclassical state \( |\psi_j\rangle \) for estimating a local phase \( \theta_j \) at the \( j \)th node. According to Eq. (4) in the manuscript, we have the uncertainty of \( \psi \rangle_{\Psi} \)

\[
\Delta^2 q \geq \sum_{l,m} w_l \left( F^{-1} \right)_{lm} w_m, \tag{1.1}
\]

where the quantum Fisher information matrix (QFIM) element for a product state is

\[
F_{lm} = 4\{ \langle \Psi | \hat{h}_l \hat{h}_m | \Psi \rangle - \langle \Psi | \hat{h}_l | \Psi \rangle \langle \Psi | \hat{h}_m | \Psi \rangle \} = 4\delta_{lm} \langle \hat{h}_m^2 \rangle \leq 4\delta_{lm} \langle \hat{h}_m^2 \rangle \equiv 4\delta_{lm} \langle \hat{n}_m^{LO} \rangle^2. \tag{1.2}
\]

We assume the resources are locally constrained such that \( n_m^{LO} \leq n \), where \( n \) does not scale with \( d \). For well-distributed weights such that \( |w|^2 \sim 1/d \), we have

\[
\Delta^2 q \geq \sum_m w_m^2 \sum_{l,m} \left( F^{-1} \right)_{lm} w_m \sim \frac{1}{n^2 \cdot d}, \tag{1.3}
\]

which shows that a product state can only give the estimation of \( q \) with a precision scaling \( \Delta q \sim 1/\sqrt{d} \) in distributed metrology. This statement holds even if the users have classical correlations between the input states. For an arbitrary input separable state \( \rho = \sum_k p_k |\Psi_k\rangle \langle \Psi_k| \), we have \( |\Psi_k\rangle = |\psi_1^{(k)}\rangle \otimes |\psi_2^{(k)}\rangle \otimes \cdots \otimes |\psi_d^{(k)}\rangle \), \( p_k \geq 0 \) and \( \sum_k p_k = 1 \). According to the convexity on the QFIM proved in the following section, we have

\[
F(\rho) \leq \sum_k p_k F(|\Psi_k\rangle \langle \Psi_k|). \tag{1.4}
\]

According to Eq. (1.2), the QFIM element for each product state satisfies \( F(|\Psi_k\rangle \langle \Psi_k|)_{lm} \leq 4\delta_{lm} \langle \hat{h}_m^2 \rangle \langle \psi^{(k)} \rangle \hat{h}_m^{(k)} | \psi^{(k)} \rangle \), therefore we obtain

\[
F(\rho) \leq \text{diag}(4 \langle \hat{h}_1^2 \rangle, \cdots, 4 \langle \hat{h}_d^2 \rangle), \tag{1.5}
\]

where \( \text{diag}\{\cdot\} \) is a diagonal matrix and \( \langle \hat{h}_m^2 \rangle \equiv \text{Tr}[\rho \hat{h}_m^2] \). Assuming locally constrained resources such that \( \langle \hat{h}_m^2 \rangle \leq n^2 \) and well-distributed weights, we obtain from Eqs. (1.1) and (1.5) that

\[
\Delta^2 q \geq \frac{1}{4n^2 \cdot d}. \tag{1.6}
\]

We have proved that without mode-entanglement the estimation precision cannot scale better than \( 1/\sqrt{d} \) in distributed metrology.
II. Convexity of the quantum Fisher information matrix

We assumed the convexity of the QFIM in deriving the estimation precision scaling for a general separable state. We are not aware of any direct proof of convexity for the QFIM in the literature, and so for completeness and internal consistency we provide a proof in this section.

**Theorem:** In multi-parameter quantum metrology, the quantum Fisher information matrix $\mathcal{F}(\rho(\theta))$ of a quantum state $\rho(\theta) = \lambda \rho_1(\theta) + (1 - \lambda) \rho_2(\theta)$, where $0 \leq \lambda \leq 1$, $\theta = (\theta_1, \ldots, \theta_d)$ and $\rho_j(\theta)$ are arbitrary quantum states, satisfies

$$\mathcal{F}(\rho(\theta)) \leq \lambda \mathcal{F}(\rho_1(\theta)) + (1 - \lambda) \mathcal{F}(\rho_2(\theta)), \quad \text{(II.1)}$$

where the QFIM is defined by its matrix element as [1]

$$\mathcal{F}_{jm}(\rho(\theta)) = \frac{1}{2} \text{Tr} \left[ \rho(\theta) \dot{L}_j \dot{L}_m + \rho(\theta) \dot{L}_m \dot{L}_j \right], \quad \text{(II.2)}$$

and the symmetric logarithmic derivative (SLD), which depends implicitly on $\rho(\theta)$, is defined via

$$\frac{\partial \rho(\theta)}{\partial \theta_l} = \frac{1}{2} \left[ \rho(\theta) \dot{L}_l + L_l \rho(\theta) \right]. \quad \text{(II.3)}$$

**Proof:** First, we define a joint state $\tilde{\rho}(\theta) = \lambda |0\rangle \langle 0| \otimes \rho_1(\theta) + (1 - \lambda) |1\rangle \langle 1| \otimes \rho_2(\theta)$, which represents a classical correlation between the probe state and a qubit ancilla with states $|0\rangle$ and $|1\rangle$. According to the monotonicity of the QFIM under completely-positive trace-preserving mapping [2], we have

$$\mathcal{F}(\tilde{\rho}(\theta)) \geq \mathcal{F}(\text{Tr}_{\text{qubit}}[\tilde{\rho}(\theta)]) = \mathcal{F}(\rho(\theta)). \quad \text{(II.4)}$$

Second, we would like to prove that $\mathcal{F}(\tilde{\rho}(\theta)) = \lambda \mathcal{F}(\rho_1(\theta)) + (1 - \lambda) \mathcal{F}(\rho_2(\theta))$, which basically follows from the fact that the two components in $\tilde{\rho}(\theta)$ are orthogonal to each other. Specifically, one can prove via the definition that a possible solution of the SLD for $\tilde{\rho}(\theta)$, denoted $\dot{L}_l$, can be written as [3]

$$\dot{L}_l = |0\rangle \langle 0| \otimes \dot{L}_l^{(1)} + |1\rangle \langle 1| \otimes \dot{L}_l^{(2)}, \quad \text{(II.5)}$$

where $\dot{L}_l^{(j)}$ is the SLD for $\rho_j(\theta)$ ($j = 1, 2$). Now inserting this result into the definition of QFIM for $\tilde{\rho}(\theta)$, we obtain

$$\mathcal{F}(\tilde{\rho}(\theta)) = \lambda \mathcal{F}(\rho_1(\theta)) + (1 - \lambda) \mathcal{F}(\rho_2(\theta)). \quad \text{(II.6)}$$

Combining Eqs. (II.4) and (II.6) proves the convexity on the QFIM.

III. Quantum Fisher information for an arbitrary separable state

The quantum Fisher information matrix element for an arbitrary input state $|\Psi\rangle$ under the unitary transformation $U$ is given by

$$\mathcal{F}_{jk} = 4 \langle \Psi | \hat{h}_j \hat{h}_k | \Psi \rangle - 4 \langle \Psi | \hat{h}_j | \Psi \rangle \langle \Psi | \hat{h}_k | \Psi \rangle, \quad \text{(III.1)}$$

where $\hat{h}_j = b_j^\dagger b_j$ are the photon number operators in the output mode. The unitary transformation $U$ relates the output mode to the input mode as $b_j^\dagger = \sum_k U_{jk} a_k^\dagger$. Therefore, the quantum Fisher information written in terms of the input modes and the initial input state is given by

$$\mathcal{F}_{jk} = 4 \langle \Psi | \sum_{l,m,r,s} U_{jl} a_l^\dagger U_{jm}^* a_m U_{kr} a_r^\dagger U_{ks}^* a_s | \Psi \rangle - 4 \langle \Psi | \sum_{l,m} U_{jl} a_l^\dagger U_{jm}^* a_m | \Psi \rangle \langle \Psi | \sum_{r,s} U_{kr} a_r^\dagger U_{ks}^* a_s | \Psi \rangle$$

$$= 4 \sum_{l,m,r,s} U_{jl} U_{jm}^* U_{kr} U_{ks}^* (\langle \hat{a}_l^\dagger \hat{a}_m \hat{a}_r \hat{a}_s \rangle - \langle \hat{a}_l^\dagger \hat{a}_m \rangle \langle \hat{a}_r^\dagger \hat{a}_s \rangle), \quad \text{(III.2)}$$

which is the expression in Eq. (7). The expression on $\mathcal{F}_{w}$ is then given by

$$\mathcal{F}_{w} = 4 \sum_{j,k} w_j w_k \sum_{l,m,r,s} U_{jl} U_{jm}^* U_{kr} U_{ks}^* (\langle \hat{a}_l^\dagger \hat{a}_m \hat{a}_r \hat{a}_s \rangle - \langle \hat{a}_l^\dagger \hat{a}_m \rangle \langle \hat{a}_r^\dagger \hat{a}_s \rangle), \quad \text{(III.3)}$$
To simplify this expression for a separable input state, we categorize the summation \(\sum_{j,m,r,s}\) into different parts depending on how many indices are identical. Hence it is expanded as

\[
F_w = 4 \sum_{jk} w_j w_k \left[ \sum_l |U_{jl}|^2 |U_{kl}|^2 \left( \langle a_j^l a_k^l a_j a_k \rangle - \langle a_j^l a_k^l \rangle^2 \right) 
+ \sum_{l \neq s} |U_{jl}|^2 U_{ks}^* \left( \langle a_j^l a_k^s a_j a_k \rangle + \langle a_j^l a_k^s a_j a_k \rangle - 2 \langle a_j^l a_k^s a_j a_k \rangle \right) + c.c. 
+ \sum_{l \neq m} U_{jl} U_{km}^* \left( \langle a_j^l a_m^s a_j a_m \rangle - \langle a_j^l a_m^s \rangle^2 \langle a_m a_m \rangle^2 \right) 
+ \sum_{l \neq m} U_{jl}^* U_{km} U_{kl} \left( \langle a_j^l a_m a_m^s a_m \rangle - \langle a_j^l a_m a_m^s \rangle \langle a_m a_m \rangle \langle a_m a_m \rangle \right) \right].
\]

(III.4)

where the terms for which \(l, m \cap r, s = \emptyset\) vanish. Defining the single-mode moments for the input states as

\[
\alpha_j \equiv \langle a_j \rangle, \quad n_j \equiv \langle a_j^2 \rangle, \quad \xi_j \equiv \langle a_j a_j^* \rangle, \quad \beta_j \equiv \langle \alpha_j \rangle, \quad m_j \equiv \langle (\alpha_j)^2 \rangle, \quad v_j \equiv m_j - n_j,
\]

(III.5)

we simplify \(F_w\) as

\[
F_w / 4 = \sum_l S_{ll}^2 v_l + \sum_{l \neq m} S_{lm} S_{ml} \left[ n_l (n_m + 1) - |\alpha_l|^2 |\alpha_m|^2 \right] 
+ \sum_{l \neq m} S_{lm} S_{ml} \left( \xi_l \xi_m - |\alpha_l|^2 |\alpha_m|^2 + \sum_{l \neq m, s} \sum_{l \neq m, s} \left( 2 n_l + 1 - 2 |\alpha_l|^2 \right) |\alpha_m|^2 + c.c. \right) .
\]

(III.6)

where \(S_{lm} = \sum_j U_{jlm} w_j U_{jls}^*\). Note that here \(n_j\), which was previously used to describe the photon number in a Fock state, is now an expectation value in a state without definite photon number. For Fock state inputs, we set \(v_l = \beta_l = \xi_l = \alpha_l = 0\) and we recover the result in Eq. (8).

Defining the terms on the right-hand side of Eq. (III.6) as \(F_j (j = 1, 2, \cdots, 6)\) respectively, we have \(F_w / 4 = F_1 + F_2 + \cdots + F_6\).

### IV. An upper bound on quantum Fisher information

Now we are going to derive the upper bound in Eq. (16) for an arbitrary separable input state. We proceed by adding positive terms to each term in the above equation. The first term is

\[
F_1 = \sum_l S_{ll}^2 v_l \leq v_{\text{max}} \sum_l S_{ll}^2 \leq v_{\text{max}} \left( \sum_l S_{ll}^2 + \sum_{l \neq m} S_{lm} S_{ml} \right) 
= v_{\text{max}} \sum_{l,m} S_{lm} S_{ml} = v_{\text{max}} \sum_l w_l^2,
\]

(IV.1)

where the first inequality holds since \(v_{\text{max}} = \max_l v_l\), the second inequality holds because \(S_{lm} S_{ml} = |S_{lm}|^2 \geq 0\) and the last equal sign is due to \(\sum_{l,m} S_{lm} S_{ml} = \sum_l w_l^2\). Similarly, the second term is

\[
F_2 = \sum_{l \neq m} S_{lm} S_{ml} \left[ n_l (n_m + 1) - |\alpha_l|^2 |\alpha_m|^2 \right] \leq M_{\text{max}} \sum_{l \neq m} S_{lm} S_{ml}
\leq M_{\text{max}} \sum_{l,m} S_{lm} S_{ml} = M_{\text{max}} \sum_l w_l^2,
\]

(IV.2)
where $M_{\text{max}} = \max_{l,m} \left[ n_l(n_m + 1) - |\alpha_l|^2 |\alpha_m|^2 \right]$. We obtain an inequality on the third term as

$$F_3 = \sum_{l,m} S^2_{ml} (\xi_l^2 \bar{e}_m - \alpha_l^2 \bar{a}_m^2) \leq \sum_{l,m} \left| S^2_{ml} (\xi_l^2 \bar{e}_m - \alpha_l^2 \bar{a}_m^2) \right|$$

$$= \sum_{l,m} S_{ml} S_{lm} \left| \xi_l^2 \bar{e}_m - \alpha_l^2 \bar{a}_m^2 \right| \leq \Xi_{\text{max}} \sum_{l,m} S_{ml} S_{lm} = \Xi_{\text{max}} \sum_{l} w_l^2,$$

where $\Xi_{\text{max}} = \max_{l,m} |\xi_l^2 \bar{e}_m - \alpha_l^2 \bar{a}_m^2|$. The steps to get an upper bound on the fourth term to the sixth term are more involved. First, we expand the fourth term in $\mathcal{F}_w/4$ as

$$F_4 = \sum_{l,m,s} S_{ml} S_{ls} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) \alpha_m \alpha_s^* - \sum_{l} S^2_{ll} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) |\alpha_l|^2$$

$$- \sum_{l,m} S_{ml} S_{lm} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) |\alpha_m|^2 - \left[ \sum_{l,m} S_{ml} S_{ll} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) \alpha_m \alpha_l^* + \text{c.c.} \right]$$

$$\leq \sum_{l,m,s} S_{ml} S_{ls} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) \alpha_m \alpha_s^* - \sum_{l,m} S_{ml} S_{ll} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) \alpha_m \alpha_l^* + \text{c.c.}$$

$$\leq \sum_{l,m,s} S_{ml} S_{ls} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) \alpha_m \alpha_s^* + 2 \sum_{l,m} S_{ml} S_{ll} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) \alpha_m \alpha_l^*,$$

where have dropped two positive-definite terms to get the first inequality and used the relation $a + a^* \leq 2|a|$ to get the second inequality. Defining $K_1 = \sum_{l,m,s} S_{ml} S_{ls} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) \alpha_m \alpha_s^*$ and $K_2 = 2 \sum_{l,m} S_{ml} S_{ll} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) \alpha_m \alpha_l^*$, then we have

$$K_1 = \sum_{l,m,s} S_{ml} S_{ls} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) \alpha_m \alpha_s^*$$

$$= \langle \alpha S^N S', \alpha \rangle \leq \langle \alpha S^N S', \alpha \rangle,$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_2d)$, $\langle \alpha, \gamma \rangle = \sum_{j} \alpha_j \gamma_j^*$ is the inner product of vectors $\alpha$ and $\gamma$, and $N'_{lm} = \delta_{ln}(2n_l + 1 - 2|\alpha_l|^2)$. To obtain an upper bound on $K_1$, we use the inequalities

$$|\langle \alpha O, \gamma \rangle| \leq ||\alpha|| \cdot \|O\| \cdot ||\gamma||,$$

$$||AB|| \leq ||A|| \cdot ||B||,$$

where || · || is the norm of a vector or a matrix. For a matrix, ||O|| is the largest eigenvalue of $O$ in the absolute value. Therefore, we have

$$K_1 \leq 2n_{\text{max}} w_{\text{max}}^2 \sum_{m} \alpha_m^2 \leq 4d n_{\text{max}} w_{\text{max}}^2 \alpha_{\text{max}}^2,$$

where $n_{\text{max}} = \max \left( n_l + 1/2 - |\alpha_l|^2 \right)$, $w_{\text{max}} \equiv \max_j |w_j|$, with $w_j$ the eigenvalues of $S$, and $\alpha_{\text{max}} = \max_l |\alpha_l|$. The last inequality from the above equation follows from $\sum_{m} \alpha_m^2 \leq 2d \alpha_{\text{max}}$. For the second term in the last line in Eq. (IV.4),

$$K_2 = 2 \sum_{l,m} S_{ml} S_{ll} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) \alpha_m \alpha_l^* - \sum_{l} S^2_{ll} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) |\alpha_l|^2$$

$$\leq 2 \sum_{l,m} S_{ml} S_{ll} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) \alpha_m \alpha_l^* + 2 \sum_{l} S^2_{ll} \left( 2n_l + 1 - 2|\alpha_l|^2 \right) |\alpha_l|^2$$

$$= 2 \langle \alpha S^N S', \alpha \rangle + 2 \langle \alpha S^2 S', \alpha \rangle$$

$$\leq 4n_{\text{max}} w_{\text{max}}^2 \sum_{m} \alpha_m^2 + 4n_{\text{max}} w_{\text{max}}^2 \sum_{m} \alpha_m^2 \leq 16n_{\text{max}} w_{\text{max}}^2 \alpha_{\text{max}}^2,$$

where $S'_{lm} = \delta_{lm} S_{ll}$ and $||S'|| \leq w_{\text{max}}$. Therefore, the fourth term in $\mathcal{F}_w/4$ is

$$F_4 \leq K_1 + K_2 \leq 20n_{\text{max}} w_{\text{max}}^2 \alpha_{\text{max}}^2.$$

(IV.10)
The fifth term in $F_{w}/4$ is expanded as

$$F_5 = \left| \sum_{l,m,s} S_{ml} S_{il} (\xi^*_l - \alpha^*_l)^2 \alpha_m \alpha_s - \sum_l S_{il} (\xi^*_l - \alpha^*_l)^2 \alpha_l^2 - \sum_{l \neq m} S_{ml} (\xi^*_l - \alpha^*_l)^2 \alpha_m^2 - 2 \sum_{l \neq m} S_{ml} S_{il} (\xi^*_l - \alpha^*_l)^2 \alpha_m \alpha_l + c.c. \right|$$

$$\leq 2 \left| \sum_{l,m,s} S_{ml} S_{il} (\xi^*_l - \alpha^*_l)^2 \alpha_m \alpha_s \right| + 2 \left| \sum_l S_{il}^2 (\xi^*_l - \alpha^*_l)^2 \alpha_l^2 \right| + 2 \left| \sum_{l \neq m} S_{ml}^2 (\xi^*_l - \alpha^*_l)^2 \alpha_m^2 \right| + 4 \left| \sum_{l \neq m} S_{ml} S_{il} (\xi^*_l - \alpha^*_l)^2 \alpha_m \alpha_l \right|.$$  

(IV.11)

We first define the terms on the right-hand side of the inequality in the above equation as $K_3, K_4, K_5, K_6$, respectively. Then we have

$$K_3 = 2 \left| \sum_{l,m,s} S_{ml} S_{il} (\xi^*_l - \alpha^*_l)^2 \alpha_m \alpha_s \right| = 2 \left| \alpha S X S, \alpha^* \right| \leq 4d \xi_{max} w_{max} \alpha_{max}^2,$$  

(IV.12)

where $X_{lm} = \delta_{lm} (\xi^*_l - \alpha^*_l)^2$ and $\xi_{max} \equiv \max_j |\xi^*_j - \alpha^*_j|$. The inequality follows from Eq. (IV.6) and Eq. (IV.7). Similarly, the second term in the above equation is

$$K_4 = 2 \left| \sum_l S_{il}^2 (\xi^*_l - \alpha^*_l)^2 \alpha_l^2 \right| = 2 \left| \alpha S^2 X, \alpha^* \right| \leq 4d \xi_{max} w_{max} \alpha_{max}^2.$$  

(IV.13)

The third term is

$$K_5 = 2 \left| \sum_{l \neq m} S_{ml}^2 (\xi^*_l - \alpha^*_l)^2 \alpha_m^2 \right| \leq 2 \xi_{max} \alpha_{max}^2 \sum_l w_l^2,$$  

(IV.14)

where the inequality is obtained using the same reasoning as in Eq. (IV.3). The fourth term on the right-hand side of the inequality in Eq. (IV.11) is

$$K_6 = 4 \left| \sum_{l \neq m} S_{ml} S_{il} (\xi^*_l - \alpha^*_l)^2 \alpha_m \alpha_l \right| \leq 4 \left| \sum_{l \neq m} S_{ml} S_{il} (\xi^*_l - \alpha^*_l)^2 \alpha_m \alpha_l \right| + 2K_4$$

$$= 4 \left| \alpha S S^* \alpha^*, \alpha^* \right| + 2K_4 \leq 16d \xi_{max} w_{max} \alpha_{max}^2.$$  

(IV.15)

Therefore, the fifth term in $F_{w}/4$ is

$$F_5 \leq K_3 + K_4 + K_5 + K_6 \leq 2 \xi_{max} \alpha_{max}^2 \sum_l w_l^2 + 24d \xi_{max} w_{max} \alpha_{max}^2.$$  

(IV.16)

Now we come to the last term in $F_{w}/4$, which is given by

$$F_6 = \sum_{l \neq s} S_{ls} S_{il} \left( 2\beta^*_l + \alpha^*_l - 2n_l \alpha^*_l \right) \alpha_s + c.c.$$  

$$\leq 2 \left| \sum_{l \neq s} S_{ls} S_{il} \left( 2\beta^*_l + \alpha^*_l - 2n_l \alpha^*_l \right) \alpha_s \right| + \sum_l S_{il} \left( 2\beta^*_l + \alpha^*_l - 2n_l \alpha^*_l \right) \alpha_l$$

$$\leq 2 \left| \sum_{l \neq s} S_{ls} S_{il} \left( 2\beta^*_l + \alpha^*_l - 2n_l \alpha^*_l \right) \alpha_s \right| + 2 \left| \sum_l S_{il} \left( 2\beta^*_l + \alpha^*_l - 2n_l \alpha^*_l \right) \alpha_l \right|$$

$$= 4 \left| \alpha S S^*, \beta^* \right| + 4 \left| \alpha S^2, \beta^* \right|.$$  

(IV.17)

where $\beta^* = (\beta_1 + \alpha_1/2 - n_1 \alpha_1, \beta_2 + \alpha_2/2 - n_2 \alpha_2, \ldots, \beta_{2d} + \alpha_{2d}/2 - n_{2d} \alpha_{2d})$. Using the inequalities in Eq. (IV.6) and Eq. (IV.7), we have

$$\left| \alpha S S^*, \beta^* \right| \leq w_{max}^2 \sqrt{\sum_l |\alpha_l|^2 \sum_m |\beta_m| + 1/2 - n_m^2} \leq 2d w_{max}^2 \alpha_{max} \beta_{max}.$$  

(IV.18)

where $\beta_{max} = \max_l |\beta_l + \alpha_l/2 - n_l \alpha_l|$. Similarly, $\left| \alpha S^2, \beta^* \right| \leq 2d w_{max}^2 \alpha_{max} \beta_{max}$. So

$$F_6 \leq 16d w_{max}^2 \alpha_{max} \beta_{max}.$$  

(IV.19)
By collecting all the terms and replacing $w_{\text{max}} = 1/d$, we finally arrive at Eq. (16) in the manuscript

$$\mathcal{F}_w \leq \frac{A}{d} + B|w|^2,$$

where

$$A = 16\alpha_{\text{max}} (5\alpha_{\text{max}} + 6\xi_{\text{max}} + 4\beta_{\text{max}}),$$

$$B = 4 \left( \alpha_{\text{max}} + \xi_{\text{max}} + 2\xi_{\text{max}}^2 \right).$$

To get a simpler bound for $\mathcal{F}_w$, we first consider the inequalities

$$n_{\text{max}} \leq \sum_j n_j + 1/2, \quad \xi_{\text{max}} \leq \max_j |\xi_j|, \quad \beta_{\text{max}} \leq \max_j |\beta_j| + \max_j n_j \max_k |\alpha_k|,$$

$$v_{\text{max}} \leq \sum_j m_j, \quad M_{\text{max}} \leq \left( \max_j n_j \right)^2 + \max_j n_j, \quad \Xi_{\text{max}} \leq \left( \max_j |\xi_j| \right)^2,$$

from the definitions of $n_{\text{max}}, \xi_{\text{max}}, \beta_{\text{max}}, v_{\text{max}}, M_{\text{max}}$ and $\Xi_{\text{max}}$. Using Cauchy-Schwarz inequality, we have

$$|\alpha_j| \leq \sqrt{n_j}, \quad n_j \leq \sqrt{m_j}, \quad |\xi_j| \leq \sqrt{m_j - n_j} \leq \sqrt{m_j}, \quad |\beta_j| \leq \sqrt{m_j n_j}.$$

Using these inequalities, we obtain an upper bound for $A$ as

$$A \leq 304 \max_j m_j + 40 \max_j n_j,$$

and an upper bound for $B$ as

$$B \leq 20 \max_j m_j + 4 \max_j n_j.$$

Note that $m_j = \langle a^\dagger_j a^\dagger_j a_j a_j \rangle + \langle a^\dagger_j a_j \rangle \geq n_j$, hence we have $A \leq 344 \max_j m_j$ and $B \leq 24 \max_j m_j$. Defining $\langle n^2 \rangle_{\text{max}} = \max_j m_j$ and for well-distributed weights $|w| \leq 1/d$, we obtain

$$\mathcal{F}_w < \frac{C^2}{d} \langle n^2 \rangle_{\text{max}},$$

where the smallest integer satisfies the above inequality is $C = 20$. Plugging the above result into Eq. (6), we arrive at Eq. (17).

V. All photons in one port

In the manuscript we have shown that the Heisenberg scaling can be achieved by consolidating all resources into just two input ports. Interestingly, in the Fock-state input case, if all of the $N$ photons are placed in a single port one cannot beat the standard quantum limit $1/\sqrt{N}$. If all the photons are placed in the first input port, for example, Eq. (III.6) is reduced to

$$\mathcal{F}_w = 4N \sum_{l=1}^N S_{1l} S_{1l} \leq 4N \sum_{l=1}^N S_{1l} S_{1l}$$

$$= 4N \sum_j w_j^2 |U_{j1}|^2 \leq 4N w_{\text{max}}^2 \leq \frac{4N}{d^2},$$

which leads to the standard quantum limit $\Delta q \geq \frac{1}{\sqrt{N}}$.