Hierarchy of linear light cones with long-range interactions

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In quantum many-body systems with local interactions, quantum information and entanglement cannot spread outside of a “linear light cone,” which expands at an emergent velocity analogous to the speed of light. Yet most non-relativistic physical systems realized in nature have long-range interactions: two degrees of freedom separated by a distance \( r \) interact with potential energy \( V(r) \propto 1/r^\alpha \). In systems with long-range interactions, we rigorously establish a hierarchy of linear light cones: at the same \( \alpha \), some quantum information processing tasks are constrained by a linear light cone while others are not. In one spatial dimension, commutators of local operators \( \langle \psi | [\hat{O}_x(t), \hat{O}_y] | \psi \rangle \) are negligible in every state \( |\psi\rangle \) when \( |x - y| \gtrsim vt \), where \( v \) is finite when \( \alpha > 3 \) (Lieb-Robinson light cone); in a typical state \( |\psi\rangle \) drawn from the infinite temperature ensemble, \( v \) is finite when \( \alpha > 5/2 \) (Frobenius light cone); in non-interacting systems, \( v \) is finite in every state when \( \alpha > 2 \) (free light cone). These bounds apply to time-dependent systems and are optimal up to subalgebraic improvements. Our theorems regarding the Lieb-Robinson and free light cones—and their tightness—also generalize to arbitrary dimensions. We discuss the implications of our bounds on the growth of connected correlators and of topological order, the clustering of correlations in gapped systems, and the digital simulation of systems with long-range interactions. In addition, we show that quantum state transfer and many-body quantum chaos are bounded by the Frobenius light cone, and therefore are poorly constrained by all Lieb-Robinson bounds.

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While non-relativistic quantum systems do not possess intrinsic absolute speed limits, their dynamics exhibit a form of causality analogous to the speed of light. Lieb and Robinson first deduced the existence of a finite velocity for the propagation of information in quantum spin systems with finite-range interactions [1]. This leads to ballistic dynamics, out of which a linear light cone emerges.

For systems with power-law interactions, i.e. those that fall off as $1/r^\alpha$ in the distance $r$ between two degrees of freedom, the story is much richer. Such long-range interactions have been exhibited in a variety of quantum simulators and technological platforms, including ultracold atomic gases [2], Rydberg atoms [3], one dimensional chains of trapped ions [4], polar molecules [5], color centers in solid-state systems [6], and atoms trapped in photonic crystals [7]. More formally, most physical systems consist of objects with electrical charges or electromagnetic dipoles, and so fundamentally these systems also exhibit long-range interactions. Hence, understanding the robustness of the properties of local physical systems in the presence of long-range interactions is essential to building and optimizing the performance of larger scale quantum technologies.

Until recently, it was unknown whether or not there existed a critical value of the power-law exponent $\alpha$ above which a linear light cone is present. Hastings and Koma [8] first demonstrated a light cone whose velocity that diverges exponentially in distance for $\alpha$ greater than the lattice dimension, $d$. Progressive improvements yielded a series of algebraic light cones for $\alpha > 2d$, which tend to a linear light cone in the limit as $\alpha \to \infty$ [9, 10]. After numerical simulations suggested the existence of a sharp linear light cone [11–13], a proof of generic linear light cones was found for systems with interaction exponent $\alpha > 2d + 1$ [14, 15].

Complementary to the Lieb-Robinson bounds are protocols that achieve the fastest allowable rates of quantum information processing. One such dynamical task is quantum state transfer, which has been used experimentally to demonstrate the transmission of entanglement in quantum systems [16]. These protocols can be directly connected to the Lieb-Robinson bound [17, 18], and have been a standard way to benchmark the sharpness of these bounds.

The goal of this paper is to answer two important questions. Firstly, is the result in Refs. [14, 15] tight, or does a universal linear light cone exist for some $\alpha < 2d + 1$? Secondly, to what extent does any optimal bound constrain practical measures of information spreading, such as quantum state transfer? In other words, are Lieb-Robinson bounds ultimately limited by physically artificial effects in systems with long-range interactions, and as such not optimal for constraining quantum information dynamics in practice?

The answer to our first question of the bound’s tightness is in the affirmative. We will demonstrate explicitly in Section 3 a dynamical protocol for which, at sufficiently small times $t$,

$$\|[[A_0(t), B_r]]\|_{\infty} \geq \frac{t^{2d+1}}{r^\alpha}.$$  \hspace{1cm} (1)

Here $A_0$ and $B_r$ represent two single-site operators a distance $r$ apart, and $A(t)$ is the version of $A$ evolved under a time-dependent Hamiltonian with two-body long-range interactions. $\|\cdot\|_{\infty}$ denotes the operator norm which returns the maximal singular value of its argument; as such, there exists a state $|\psi\rangle$ in which $|\langle\psi|[A_0(t), B_r]|\psi\rangle| \geq t^{2d+1}/r^\alpha$. The existence of this protocol rules out any “Lieb-Robinson light cones” of the form $t \geq r^\kappa$ with $\kappa > \alpha/(2d + 1)$ and in particular rules out any possible generalizations of the linear light cones of Refs. [14, 15] to the regime $\alpha < 2d + 1$; in other words, we prove the sharpness of these recent bounds. After tightening the recent bound of Ref. [15] when applied to multi-site operators, we discuss applications of this (nearly) optimal Lieb-Robinson light cone to the growth of correlation functions, to digital simulation of quantum systems, and to ground state correlation functions.

Surprisingly, the answer to our second question is that the Lieb-Robinson light cone is often saturated by finely tuned protocols, and that practical information-spreading tasks—such as state transfer using few-body interactions—are controlled by a linear light cone even when the Lieb-Robinson light cone is not linear. To be specific, we consider...
FIG. 1. The hierarchy of linear light cones in one dimension; we say that a light cone has exponent $\gamma$ if $\| [A_0(t), B_r] \|$ is large only when $t \gtrsim r^\gamma$. The plot depicts the exponents of the Lieb-Robinson light cone (solid line) [14], the Frobenius light cone from Theorem 7 (dot-dashed line), and the free light cone from Theorem 9 (dashed line) as functions of $\alpha$ in one dimension. The free light cone is known to be a tight bound for all $\alpha$. We also show that the Lieb-Robinson and Frobenius light cones are not linear below $\alpha = 3$ and $\alpha = \frac{5}{2}$ respectively.

the Frobenius operator norm

$$\| [A_0(t), B_r] \|_F := \sqrt{\frac{\text{tr}( [A_0(t), B_r]^\dagger [A_0(t), B_r] )}{\text{tr}(1)}}.$$ (2)

This operator norm can be interpreted as the out-of-time-ordered correlation function used to probe early time chaos in many-body systems [19, 20] or, equivalently, as the “fraction” of the operator $A_0(t)$ that has support on the site $r$. We prove in Section 4 that $\| [A_0(t), B_r] \|_F$ is bounded inside of a linear light cone in one-dimensional models with two-body interactions so long as $\alpha > \frac{5}{2}$, and also demonstrate the optimality of this bound, up to subalgebraic corrections. This “Frobenius light cone” has a number of important consequences. Firstly, many-body quantum chaos is not constrained effectively by the Lieb-Robinson light cone; in more practical terms, the dynamical protocols that saturate any Lieb-Robinson bounds can only have consequences in finely tuned quantum states, while on typical states these protocols become ineffective. Moreover, we show that quantum state transfer is constrained by this stronger Frobenius light cone. Contrary to prior expectations, fundamentally new mathematical frameworks are required to obtain effective bounds on state transfer and entanglement generation in generic quantum systems. We conjecture that in the general case of $d$ dimensions, the Frobenius light cone is linear when $\alpha > \frac{3}{2}d + 1$.

In Section 5, we obtain a third light cone for systems that are described by non-interacting bosons or fermions. In these systems, we prove that $\| [A_0(t), B_r] \|$ is constrained by a linear light cone when $\alpha > d + 1$, and by a “superlinear” light cone of the form $t \sim r^{\alpha - d - \epsilon}$, for any $\epsilon > 0$, when $d < \alpha \leq d + 1$. We show that both our linear and superlinear light cones are tight up to subalgebraic corrections by presenting a quantum state-transfer protocol for a non-interacting system that (nearly) saturates our bounds. Remarkably, our single particle state-transfer protocol performs asymptotically as well as the state-transfer protocol of Ref. [17], which only applies to interacting systems. We also use the free light cone to bound the simulation of certain quantum systems, including the Bose-Hubbard model at low densities, a problem that arose as a candidate for the demonstration of quantum supremacy [21]. We illustrate the Lieb-Robinson, the Frobenius, and the free light cones for one-dimensional systems in Fig. 1.

Finally, we present two brief applications of Lieb-Robinson bounds. In Section 6, we prove that long-range interactions cannot parametrically speed up the preparation of topologically ordered states until $\alpha \leq 3d + 1$, and in Section 7 we describe the decay of spatial correlators in the ground state of gapped Hamiltonians with long-range interactions.

2. PRELIMINARIES

We now more carefully introduce the problem that we address in this paper. First, we will give a precise definition of a many-body quantum system with long-range interactions; then we will give the heuristic explanation for the hierarchy of three light cones highlighted above.
2.1. Long-range interactions

In order to discuss long-range interactions, we need to first define the distance between two points. Formally, we do so as follows. Let \( \Lambda \) be the vertex set of a \( d \)-dimensional lattice graph with edge set \( E_\Lambda \). A lattice graph \((\Lambda, E_\Lambda)\) is a graph which is invariant under \( d \)-dimensional discrete translations: mathematically speaking, \( \mathbb{Z}^d \subseteq \text{Aut}(\Lambda, E_\Lambda) \), where \( \text{Aut} \) denotes the group of graph isomorphisms from \((\Lambda, E_\Lambda)\) to itself. We assume that all vertices have finite degree in \( E_\Lambda \), and that \( |\Lambda/\mathbb{Z}^d| < \infty \), i.e. the unit cell has a finite number of vertices, and every vertex has a finite number of (nearest) neighbors. This graph imbues a natural notion of distance, which we will use for the rest of the paper. Let \( D: \Lambda \times \Lambda \to \mathbb{Z}^+ \) denote the shortest path length between two vertices, also known as the Manhattan metric.

A many-body quantum system is then defined by placing a finite-dimensional quantum system (e.g. a qubit) on every vertex in \( \Lambda \). Formally we define a many-body Hilbert space
\[
H := \bigotimes_{i \in \Lambda} H_i,
\]
where we assume that \( \dim(H_i) < \infty \). In this paper, we will be especially interested in the dynamics of the operators acting on \( H \). Let \( \mathcal{B} \) denote the set of all Hermitian operators acting on \( H \). \( \mathcal{B} \) is a real vector space, and we denote operators \( O \in \mathcal{B} \) with \( |O \rangle \) whenever we wish to emphasize that they should be thought of as vectors. A basis for \( \mathcal{B} \) can be found as follows: let \( T^a_i \) denote the generators of \( \text{U}(\dim(H_i)) \) where \( a = 0 \) denotes the identity operator, which gives a complete basis for Hermitian operators on the local Hilbert space \( H_i \). \( \mathcal{B} \) is simply the tensor product of all these local bases of Hermitian operators:
\[
\mathcal{B} = \text{span} \left\{ \bigotimes_{i \in \Lambda} T^a_i, \text{ for all } \{a_i\} \right\}.
\]

For subset \( X \subset \Lambda \), we define \( \mathcal{B}_X \) to be the set of all basis vectors which act non-trivially only on the sites of \( X \):
\[
\mathcal{B}_X := \text{span} \left\{ \bigotimes_{i \in X} T^a_i, \text{ for all } \{a_i \neq 0\} \right\}.
\]

We define the projectors \( P_i \) by
\[
P_i \otimes T^a_k := \begin{cases} | \otimes T^a_k & a_i \neq 0 \\ 0 & a_i = 0 \end{cases},
\]
which return the part of the operator that acts non-trivially on site \( i \):
\[
P_i O = O - \frac{1}{\dim(H_i)} \text{tr}_i O.
\]

For a general subset \( X \subset \Lambda \), the projectors
\[
P_X := \sum_{Y \in \mathbb{Z}_2^d : |Y| > 0} (-1)^{|Y|+1} \prod_{j \in Y} P_j
\]
act similarly, and return the part of the operator which acts non-trivially on the subset \( X \). It was proven in Ref. [14] that when \( |X| < \infty \),
\[
\|P_X O\|_\infty \leq 2\|O\|_\infty,
\]
where \( \|\cdot\|_\infty \) is again the operator norm. We will often drop the \( \infty \) subscript for convenience. In addition, we can relate the commutator in the Lieb-Robinson bound to the projection of an operator using the identity
\[
\|[O_X, O_Y]\| \leq 2\|O_X\|\|P_X O_Y\|,
\]
which holds for all operators \( O_X \in \mathcal{B}_X, O_Y \in \mathcal{B}_Y \).

We define the Hamiltonian \( H: \mathbb{R} \to \mathcal{B} \) as
\[
H(t) := \sum_{X \subset \Lambda} H_X(t),
\]
where $H_X(t) : \mathbb{R} \to \mathcal{B}_X$. $H(t)$ is said to be $q$-local if $H_X(t) = 0$ for all $|X| > q$: physically speaking, the Hamiltonian operator contains at most $q$-body interactions. The Hamiltonian generates time evolution on $\mathcal{B}$ according to the Heisenberg equation of motion for operators: we define the Liouvillian $\mathcal{L}(t)$ as the generator of time evolution,

$$\mathcal{L}(t) |\mathcal{O}| := \{ [H(t), \mathcal{O}](t) \},$$

We define the time evolved operator $\mathcal{O}(t) : \mathbb{R} \to \mathcal{B}$ as the solution to the differential equation

$$\frac{d\mathcal{O}(t)}{dt} := \mathcal{L}(t) \mathcal{O}(t), \quad \mathcal{O}(0) := \mathcal{O}.$$  

We say that the Hamiltonian $H$ has long-range interactions with exponent $\alpha$ if

$$\alpha = \sup \left\{ \alpha_0 \in (0, \infty) : \text{there exists } 0 < h < \infty \text{ such that } \sum_{X:|i,j| \leq X} \|H_X(t)\| \leq \frac{h}{D(i,j)^{\alpha_0}}, \text{ for all } t \in \mathbb{R} \right\},$$

where $D(i,j)$ denotes the distance between $i,j$. In physics we often say that the interaction has exponent $\alpha$ when, assuming only two-body interactions, $H_{i,j} \lesssim h D(i,j)^{-\alpha}$; strictly speaking though, any Hamiltonian with exponent $\alpha_2$, according to this loose definition, also has exponent $\alpha_1 < \alpha_2$. The formal definition Eq. (14) avoids this unwanted feature and assigns a unique exponent $\alpha$ to every problem.

The following identities, which we state without proof, will be useful in the discussion that follows:

**Proposition 1** (Sums over power laws \[8, 10\]). If $\alpha > d$, for any $\Lambda$ and $\mathcal{D}$, there exist $0 < C_1, C_2 < \infty$ such that:

$$\sum_{j \in \Lambda: D(i,j) > r} \frac{1}{D(i,j)^{\alpha}} < \frac{C_1}{r^{\alpha-d}},$$

$$\sum_{k \in \Lambda \setminus \{i,j\}} \frac{1}{D(i,k)^{\alpha} D(j,k)^{\alpha}} < \frac{C_2}{D(i,j)^{\alpha}}.$$  

**2.2. Heuristic arguments**

Now that we have formally defined the problem, we make a few heuristic arguments as to why the three light cones we identified in the introduction become parametrically separated in the presence of long-range interactions. In the arguments below, we assume the Hamiltonian is 2-local.

We begin with the Lieb-Robinson light cone, which is controlled by the growth of the operator norm of a commutator $\|[A_0(t), B_r]\|$. Let us consider a scenario where $A_0(t)$ evolves to a large product operator of the form

$$A_0(t) = \bigotimes_{j \in \Lambda: D(j,0) < r/2} A_j.$$  

With local interactions alone, this takes a time $t \propto r$. Now we ask how long it takes to use long-range interactions to push the operator onto sites a distance at least $r$ away, a task that we achieve via Hamiltonian

$$H' = \sum_{i \in \Lambda: D(i,0) < \frac{r}{2}} \sum_{j \in \Lambda: D(j,0) \geq r} \frac{O_{ij}}{D(i,j)^{\alpha}}, \text{ where } \|O_{ij}\| = 1.$$  

Letting $\mathcal{P}_{r}$ project onto the operators that act on sites $\geq r$, we estimate

$$\frac{\|\mathcal{P}_{r} e^{iH't'} A_0(t) e^{-iH't'}\|}{\|A_0(t)\|} \leq 2t'\|H', A_0(t)\| \leq 4t'\|H'\| \leq 4t' \left( \sum_{i \in \Lambda: D(i,0) < \frac{r}{2}} \sum_{j \in \Lambda: D(j,0) \geq r} \frac{1}{D(i,j)^{\alpha}} \right) = O\left( \frac{t'}{r^{\alpha-2d}} \right).$$  

Once $\alpha < 2d + 1$, it becomes faster to use the long-range interactions to grow the operator. In other words, we can only expect linear Lieb-Robinson light cones for $\alpha > 2d + 1$. See Figure 2 for a depiction of this effect.
Hence the magnitude of this term has decreased from α to d. Thus, a single particle to go a distance at least r final site j is linear above α > d. We assume H only contains two-body interactions. The critical values of α after which this norm becomes large differs depending on whether we use the operator or Frobenius norm, and whether the system is interacting or free. Note that the scaling with r does not change if B₂ is replaced by U₂, the region outside the dotted line which includes every site a distance ≥ r from B₁.

If we use the Frobenius norm instead of the operator norm in Eq. (19), more care is required. Let us study more closely the commutator

\[ [A₀(t), H'] = \sum_{j ∈ \Lambda : \mathcal{D}(j, 0) ≥ r} \left[ \sum_{i ∈ \Lambda : \mathcal{D}(i, 0) < d/2} \frac{O_{ij}}{D(i, j) \alpha} A₀(t) \right]. \tag{20} \]

The outermost sum represents a sum over operators which are each supported on distinct subsets of \( \Lambda \)–each term that survives this sum acts on a distinct site \( j \). Upon squaring this operator and taking a trace, cross terms from different sites \( j \) and \( j' \) will vanish. Hence,

\[
\left\| \mathbb{P}_{≥r} e^{iH't'} A₀(t)e^{-iH't'} \right\|_F ≤ t' \left\| [H', A₀(t)] \right\|_F \leq 2t' \sqrt{\sum_{j ∈ \Lambda : \mathcal{D}(j, 0) ≥ r} \left\| \sum_{i ∈ \Lambda : \mathcal{D}(i, 0) < d/2} \frac{2O_{ij}}{D(i, j) \alpha} \right\|^2} \leq 4t' \sqrt{\sum_{j ∈ \Lambda : \mathcal{D}(j, 0) ≥ r} \frac{1}{D(j, 0)^{2(\alpha - d)}}} \lesssim O\left(\frac{t'}{r^{\alpha - 3d/2}}\right). \tag{21} \]

Hence the magnitude of this term has decreased from \( \alpha - 2d \) to \( \alpha - \frac{3d}{2} \); as such, we expect that the Frobenius light cone is linear above \( \alpha > \frac{3}{2}d + 1 \). Note that the sum over initial site \( i \) in Eq. (21) adds up linearly, while the sum over final site \( j \) adds in squares (with an overall square root).

Lastly, if we have a system of non-interacting particles, it suffices to follow the motion of a single particle. The hopping rate of a single particle to go a distance at least \( r \) is constrained in the worst case by

\[
\left\| \sum_{j ∈ \Lambda : \mathcal{D}(j, 0) ≥ r} \frac{c_j c₀}{D(j, 0)^\alpha} \right\| ≤ O\left(\frac{1}{r^{\alpha - d}}\right). \tag{22} \]

where \( c^\dagger \) and \( c \) are the creation and the annihilation operators, and so the free particle is constrained within a linear light cone when \( \alpha > d + 1 \). In Section 5.4, we will show that this estimated growth rate is indeed obtained by a coherent superposition of a single particle wave function on many sites.

### 3. Lieb-Robinson Light Cone

Having given heuristic arguments for the hierarchy of light cones, we now return to a precise discussion. We begin by presenting the strictest light cone on the commutators of local operators, representing the generalization of the Lieb-Robinson theorem [1] to systems with long-range interactions.
3.1. The linear light cone

The following proposition controls the growth of commutator norms in a Hamiltonian system with long-range interactions.

**Proposition 2.** Let $X, Y \subseteq \Lambda$ be disjoint with $D(X, Y) := r; O_X$ be an operator supported on $X$ obeying $\|O_X\| = 1$; $O_X(t)$ be the time-evolved version of $O_X$ under a power-law Hamiltonian with an exponent $\alpha > 2d + 1$. There exist constants $0 < \bar{v}, c < \infty$ such that, for time evolutions generated by Eq. (12) obeying Eq. (14),

$$\|\mathbb{P}_Y |O_X(t)\| \leq c |X| \frac{\bar{v}^{d+1} \log^{2d} r}{(r - \bar{v} t)^{\alpha-d}}.$$  \hspace{1cm} (23)

**Proof.** We begin by recalling the following theorem (recast in the language of projectors):

**Theorem 3** (Linear light cone [15]). Eq. (23) holds for a single-site operator, i.e. when $|X| = 1$.

While the proof presented in Ref. [15] applied only to time-independent Hamiltonians, the generalization to time-dependent models is immediate from their results. Next, we show the following general result.

**Lemma 4.** If for all $x \in X$, $\|\mathbb{P}_Y |O_x(t)\| \leq f(t, D(r, Y))$, then there exist $0 < K < \infty$ such that

$$\|\mathbb{P}_Y |O_X(t)\| \leq K \sum_{x \in X} f(t, D(x, Y)).$$  \hspace{1cm} (24)

**Proof.** For pedagogical reasons, we demonstrate the proof on a system of spin-1/2 particles with $K = 9/2$. However, the proof applies to any system with finite local Hilbert space dimensions [14]. Let $\{S_j : j = 1, \ldots, d_Y - 1\}$ denote the $d_Y - 1 = 4^{|Y|} - 1$ nontrivial Pauli strings supported on $Y$. Then [14]

$$\|\mathbb{P}_Y |O_X(t)\| = \left\| \frac{1}{2d_Y} \sum_{j=1}^{d_Y-1} [S_j, S_j, O_X(t)] \right\| \leq \frac{1}{2d_Y} \sum_{j=1}^{d_Y-1} 2 \|S_j\| \|[S_j, O_X(t)]\|$$

$$\leq \frac{1}{d_Y} \sum_{j=1}^{d_Y-1} \|O_X, S_j(-t)\| \leq \frac{2}{d_Y} \sum_{j=1}^{d_Y-1} \|\mathbb{P}_X |S_j(-t)\|.$$  \hspace{1cm} (25)

Next, we shall prove that

$$\|\mathbb{P}_X |S_j(-t)\| \leq 3 \sum_x \|\mathbb{P}_x |S_j(-t)\|.\hspace{1cm} (26)$$

To do so, we assign an (arbitrary) ordering of the sites in $X$: i.e. if $X = \{x_1, \ldots, x_n\}$, we choose $x_1 < x_2 < \cdots < x_n$. Let $\bar{X}_x = \{x' \in X : x' > x\}$ be a subset of $X$ consisting of sites in $X$ that are greater than $x$. We rewrite

$$\mathbb{P}_X = \sum_x (1 - \mathbb{P}_{\bar{X}_x}) \mathbb{P}_x,$$  \hspace{1cm} (27)

and therefore we have

$$\|\mathbb{P}_X |S_j(-t)\| \leq \sum_x \|\mathbb{P}_x |S_j(-t)\| \leq \sum_x 3 \|\mathbb{P}_x |S_j(-t)\|.$$  \hspace{1cm} (28)

In the last line, we have used that $\|\mathbb{P}_X O\| \leq 2\|O\|$ whenever $|X| < \infty$ [14], along with the triangle inequality. Plugging this back into the earlier equation, we have

$$\|\mathbb{P}_Y |O_X(t)\| \leq \frac{6}{d_Y} \sum_{j=1}^{d_Y-1} \sum_x \|\mathbb{P}_x |S_j(-t)\| \leq \frac{6}{d_Y} \sum_{j=1}^{d_Y-1} \sum_x \frac{1}{8} \sum_{P_x} \|[P_x, [P_x, S_j(-t)]]\|\$$

$$\leq \frac{3}{2} \frac{1}{d_Y} \sum_{j=1}^{d_Y-1} \sum_x \sum_{P_x} \|\mathbb{P}_{S_j} P_x(t)\| \leq \frac{9}{2} \sum_x f(t, D(x, Y)),\hspace{1cm} (29)$$

where $P_x \in \{X_x, Y_x, Z_x\}$ denotes one of the three Pauli matrices on site $x$. In the second from the last line, we have used the assumption $\|[P_x, P_y(t)]\| \leq f(t, D(x, Y))$.  

Combining Theorem 3 with Lemma 4 proves Eq. (23), which is tighter than a result of Ref. [15] when applied to general operators that are supported on many sites.
FIG. 3. A protocol for rapid growth of the commutator norm using two-body long-range interactions. Step 1: we use CNOT gates between nearest neighbor sites to spread a single Pauli $X_0$ to a Pauli string $XX \cdots X$ supported on every site inside a ball of radius $O(t)$ centered at $X_0$. Step 2: we use pairwise $ZZ$ interactions between all sites in the two balls, located distance $O(r)$ apart, which adds an operator of norm $\sim O(t^{2d+1}/r^\alpha)$ into the second ball a distance $r$ away. Step 3: we invert Step 1 in the outer ball, pushing all of the operator weight in the outer ball onto a single site.

### 3.2. Fast operator spreading protocol

Proposition 2 proves that the support of an operator $O_i(t)$ is only large inside of a linear light cone when $\alpha > 2d + 1$. Our first main result is the following theorem, which proves the optimality (up to subalgebraic corrections) of that result.

**Theorem 5.** Let $\dim(H_i) = 2$ for all $i \in \Lambda$, and let $X_0$ and $X_r$ be two Pauli-$X$ operators supported on two sites $i$ and $j$ respectively, obeying $D(i,j) = r$. For all $\alpha > d$, there exists a time-dependent Hamiltonian $H(t)$ obeying Eq. (14) and constants $0 < K, K' < \infty$ such that for $3 < t < K'r^{\alpha/(1+2d)}$, 

$$\| [X_0(t), X_r] \| \geq Kt^{1+2d}/r^\alpha.$$  

(30)

**Proof.** We prove the theorem by constructing a fast operator-spreading protocol, which follows three steps, as depicted in Figure 3. In each step, we evolve the operator using a power-law Hamiltonian for time $t/3$. For simplicity, we assume $t/3 := \ell \in \mathbb{Z}^+$, and assume that $\ell < \frac{1}{2}r$.

**Step 1.** In time $t/3$, we use a unitary $U_1$ to spread the operator $X_0$ to $\prod_{i \in B_\ell} X_i$, where $B_\ell$ is a ball of radius $\ell$ centered at site 0. We denote the volume of this ball by $V := |B_\ell|$. The unitary $U_1$ can be implemented using a series of controlled-NOT operators (CNOT) among nearest neighbors in the lattice. Note that a CNOT gate $U_{\text{CNOT},i,j}$ for neighbors $i$ and $j$ acts as follows:

$$U_{\text{CNOT},i,j}^\dagger X_i U_{\text{CNOT},i,j} := X_i X_j.$$  

(31)

Under the conditions of Eq. (14), this CNOT gate can be implemented in a time step of $O(1)$.

**Step 2.** In the next $t/3$ interval, we apply $U_2 = \prod_{j \in B_\ell} U_j(\tau)$ on the operator, where

$$U_j(\tau) := \cos(\tau \Theta) + i \sin(\tau \Theta) Z_j,$$

$$\Theta := \sum_{k \in B_\ell} Z_k,$$  

(32a)

(32b)
\[ B_r \] is another ball of radius \( d \) centered the site at the distance \( r \), and

\[ \tau = \frac{t}{3(2r)^a}. \]  

(33)

It is straightforward to verify that \( U_j(\tau) \) is a unitary, since

\[ U_j(\tau) = \exp \left[ -i\tau \sum_{y \in B_t} Z_y \right]. \]  

(34)

Since \( Z_y \) commutes with \( Z_{y'} \) for all \( j, j' \in B_t \) and \( y, y' \in \tilde{B}_t \), \( U_j(\tau) \) and \( U_{j'}(\tau) \) can be implemented simultaneously. In other words, the unitary \( \mathcal{U}_2 \) can be generated by a power-law Hamiltonian within time \( t/3 \): the factor of \( 2r \) in Eq. (33) is present because the maximal distance between two sites in \( B_t \) and \( \tilde{B}_t \) is \( r + 2\ell < 2r \). The evolved version of the operator under this unitary is

\[ \mathcal{U}_2^{\dagger} \left( \prod_{j \in \mathcal{B}_t} X_j \right) \mathcal{U}_2 = \prod_{j \in \mathcal{B}_t} \left[ \cos(2\tau \Theta)X_j + \sin(2\tau \Theta)Y_j \right]. \]  

(35)

Step 3—. In the final \( t/3 \), we apply a unitary \( \mathcal{U}_3 \) which is the inverse of \( \mathcal{U}_2 \), up to its action on \( \tilde{B}_t \) instead of \( B_t \). It is easier to instead think of evolving the final operator \( X_r \) under \( \mathcal{U}_3^{-1} \), which does not change the commutator norm \( \| [X_0(t), X_r] \| \). Therefore, after time \( t \), we get the commutator norm:

\[ \| [X_1(t), X_r] \| = \| \left[ \mathcal{U}_3^{\dagger} X_0 \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 X_r \mathcal{U}_3^{\dagger} \right] \| = \| \left[ \prod_{j \in \mathcal{B}_t} \left[ \cos(2\tau \Theta)X_j + \sin(2\tau \Theta)Y_j \right], \prod_{k \in \tilde{B}_t} X_k \right] \| \equiv C. \]  

(36)

To lower bound the norm of \( C \), we consider the matrix elements of \( C \) in the eigenbasis of Pauli \( Z \) operators. We observe that \( \langle e | C | 00 \ldots 0 \rangle = 0 \) for all computational basis states |e\rangle of the two balls except for |e⟩ = |11...1⟩. Hence,

\[ \langle 11...1 | C | 00...0 \rangle = \left[ \cos(2\tau V) - i \sin(2\tau V) \right]^V - \left[ \cos(2\tau V) + i \sin(2\tau V) \right]^V = -2 \sum_{k \text{ odd}}^V \binom{V}{k} i^k \sin(2\tau V)^k \cos(2\tau V)^{V-k} := a. \]  

(37)

Therefore, \( C \) is block diagonal and has eigenvalues \( \pm |a| \) in the sector \{00...0\}, \{11...1\}. Thus, we can bound the norm of \( C \):

\[ \| C \| \geq |a| \geq 2V \sin(2\tau V) \cos(2\tau V)^{V-1} - \sum_{k \text{ odd}, k \geq 3}^V \binom{V}{k} \sin(2\tau V)^k \cos(2\tau V)^{V-k} \geq 2V \sin(2\tau V) \cos(2\tau V)^{V-1} - \frac{V^3}{6} \sin(2\tau V)^3 \sum_{k \text{ even}}^{V-3} \binom{V-3}{k} \sin(2\tau V)^k \cos(2\tau V)^{V-3-k} \geq 2V \sin(2\tau V) \cos(2\tau V)^{V-1} - \frac{V^3}{6} \sin(2\tau V)^3 [\sin(2\tau V) + \cos(2\tau V)]^{V-3}. \]  

(38)

Now we require that \( V^2 \tau = \epsilon < 1/2 \), which is equivalent to \( t^{2d+1} \lesssim \tau^a \). Under this condition,

\[ \cos(2\tau V)^{V-1} \geq (1 - \tau^2 V)^V \geq \left( 1 - \frac{\epsilon^2}{V^2} \right)^V \geq \frac{1}{2^V} \]  

(for all \( V \geq 1 > \epsilon^2/10 \),

(39a)

\[ [\sin(2\tau V) + \cos(2\tau V)]^{V-3} \leq (1 + 2\tau V)^V \leq \left( 1 + \frac{2\epsilon}{V} \right)^V \leq e^{2\epsilon}, \]  

(39b)

\[ \tau V \leq \sin(2\tau V) \leq 2\tau V. \]  

(39c)
Therefore,
\[
\|C\| \geq 2\mathcal{V}(\tau) \left[ \frac{1}{2} \frac{\mathcal{V}^2}{12} \left( 2\tau \mathcal{V} \right)^2 e^{2\tau} \right] \geq \mathcal{V}^2 \tau \left( 1 - \frac{2}{3} e^{2\tau} \right) \geq \frac{1}{2} \mathcal{V}^2 \tau
\]
\[
\geq \frac{1}{2} \left( \frac{1}{3\mathcal{V}} \right)^2 \frac{t^{1/2} \epsilon}{1+2d/3} \geq \frac{1}{2} \frac{t^{d+1}}{3^{1+2d/1+\alpha}} \frac{\epsilon^{2d+1}}{r^\alpha}. \tag{40}
\]
This protocol shows that if the light cone of a Lieb-Robinson bound is \( t \geq r^\kappa \), then \( \kappa \leq \alpha/(1+2d) \).

Lastly, we note that it is trivial to remove the restriction \( \dim(\mathcal{H}_t) = 2 \) from the assumptions of Theorem 5 by simply making \( H(t) \) act trivially on all but 2 of the basis states in each \( \mathcal{H}_t \).

### 3.3. Growth of connected correlators

In this subsection, we explore how fast connected correlators can be generated using a power-law Hamiltonian. In particular, we use the Lieb-Robinson bounds to show that the growth of connected correlators is constrained to linear light cones for all \( \alpha > 2d + 1 \). In contrast, for \( \alpha < 2d + 1 \), we construct—a based on our protocol in Theorem 5—an explicit example where the growth of connected correlators is not contained within any linear light cone.

We consider a \( d \)-dimensional lattice \( \Lambda \) and a power-law Hamiltonian \( H(t) \) with an exponent \( \alpha \). Let \( C \) denote a plane that separates \( \Lambda \) into two disjoint subsets \( L \) and \( R \), with \( L \cup R = \Lambda \). Let \( A \) and \( B \) be two unit-norm operators supported on single sites \( x \in L \) and \( y \in R \) respectively such that \( D(x,C), D(y,C) > r/2 \). Finally, let \( \langle \psi \rangle \) be a product state between the sublattices \( L, R \). Our object is the connected correlator
\[
C(t, r) = \langle A(t) B(t) \rangle - \langle A(t) \rangle \langle B(t) \rangle,
\]
where \( \langle \cdot \rangle = \langle \psi | \cdot | \psi \rangle \) and \( A(t) \) is the time-evolved version of \( A \) under \( H \). While the correlator vanishes at time zero due to the disjoint supports of \( A \) and \( B \), it may grow with time as the operators spread under the evolution.

First, we show that \( C(t, r) \) obey a linear light cone for all \( \alpha > 2d + 1 \). Our strategy is to approximate \( A(t) \) by an operator \( \tilde{A} \) supported on a ball of radius \( r/2 \) centered on \( x \) and \( B(t) \) by \( \tilde{B} \) supported on a ball of the same radius but centered on \( y \). Since \( \tilde{A} \) and \( \tilde{B} \) have disjoint supports, the connected correlator between them vanishes. Therefore, the connected correlator between \( A(t) \) and \( B(t) \) is upper bounded by the errors of the approximations:
\[
C(t, r) \leq \left\| A(t) - \tilde{A} \right\| + \left\| B(t) - \tilde{B} \right\|,
\]
which in turn depend on the constructions of \( \tilde{A}, \tilde{B} \).

Let \( S_A \) contain sites that are at most a distance \( r/2 \) away from \( x \) and \( S_A^c \) be all other sites in the lattice. We construct \( \tilde{A} \) by simply tracing out the part of \( A(t) \) that lies outside \( S_A \) [23]:
\[
\tilde{A} = \text{tr}_{S_A^c} [A(t)],
\]
where the partial trace is taken over \( S_A^c \). It follows from the definition that \( \tilde{A} \) is supported entirely on \( S_A \). Proposition 2 provides a bound on the error in approximating \( A(t) \) by \( \tilde{A} \): there exists a velocity \( u \) such that when \( r > ut \),
\[
\left\| A(t) - \tilde{A} \right\| \leq K \frac{t^{d+1} \log^{2d} r}{r^{\alpha - d}}, \tag{44}
\]
for some constant \( 0 < K < \infty \). Upper bounding the error in approximating \( B(t) \) by \( \tilde{B} \) in a similar way, we obtain a bound on the connected correlator:
\[
C(t, r) \leq 2K \frac{t^{d+1} \log^{2d} r}{r^{\alpha - d}}. \tag{45}
\]
As a result, the light cone of the connected correlator is linear, with velocity no larger than \( u \), for \( \alpha > 2d + 1 \).

We now provide an example of superlinear growth of connected correlators for \( \alpha < 2d + 1 \) using a slightly altered protocol than that discussed in Section 3.2. In particular, we consider initial operators \( A = X_x \) and \( B = Z_y \) supported on \( x, y \) respectively such that \( D(x,y) = r \).

The protocol works as follows. In the first step of the protocol (again in time \( t/3 \)), we still apply \( \mathcal{U}_1 \) in order to spread \( X_x \) to \( \prod_{i \in B_x} X_i \), where \( B_x \) is a ball of radius \( \ell = t/3 \) centered on \( x \). Since \( \mathcal{U}_1 \) acts trivially on \( B_x \) (the ball of
radius ℓ centered on y), we can simultaneously apply a locally rotated version of $\mathcal{U}_t$ in $\tilde{\mathcal{B}}_t$ to spread $Z_y$ to $\prod_{i \in \tilde{\mathcal{B}}_t} Z_i$. In the next time $t/3$, we again apply $\mathcal{U}_2$, which takes $\prod_{i \in \tilde{\mathcal{B}}_t} X_i$ to the expression in Eq. (35). Note that this evolution does not change $\prod_{i \in \tilde{\mathcal{B}}_t} Z_i$ as it commutes with $\mathcal{U}_2$. For the last $t/3$ we simply do nothing.

Define the state $|\psi\rangle = |\phi\rangle_{\mathcal{B}_t} |\phi\rangle_{\mathcal{B}_t}$, where

$$
|\phi\rangle_{\mathcal{B}_t} = \frac{1}{\sqrt{2}} \left( |0 \cdots 0\rangle_{\mathcal{B}_t} + i |1 \cdots 1\rangle_{\mathcal{B}_t} \right)
$$

(46)

is a state of the sites in $\mathcal{B}_t$. Throughout our analysis, we will often dispense with the subscripts, but the Hilbert space in question should be clear from context, and we will always list the state on $\mathcal{B}_t$ before that for $\mathcal{B}_t$.

We will calculate the connected correlator

$$
C(t, r) = \langle Z_y(t) X_x(t) \rangle - \langle Z_y(t) \rangle \langle X_x(t) \rangle,
$$

(47)

where $\langle \cdot \rangle = \langle \psi | \cdot | \psi \rangle$ and $X_x(t), Z_y(t)$ are the operators evolved under the unitaries described above. Assume for simplicity that $t$ is such that $V$—the volume of $\mathcal{B}_t$—is odd. It is straightforward to show that $\langle Z_y(t) \rangle = 0$ and therefore the second term $C(t, r)$ vanishes. Next, we have

$$
X_x(t) |\psi\rangle = \prod_{j \in \mathcal{B}_t} \left[ \cos(2\tau \Theta) X_j + \sin(2\tau \Theta) Y_j \right] |\psi\rangle
$$

$$
= \frac{1}{\sqrt{2}} \prod_{j \in \mathcal{B}_t} \left[ c X_j + s Y_j \right] |\phi\rangle |\bar{0}\rangle + \frac{1}{\sqrt{2}} \prod_{j \in \mathcal{B}_t} \left[ c X_j - s Y_j \right] |\phi\rangle |\bar{1}\rangle
$$

$$
= \frac{1}{2} \left[ (c + i s) V |\bar{1}\rangle |\bar{0}\rangle + i(c - i s) V |\bar{0}\rangle |\bar{1}\rangle + i(c - i s) V |\bar{1}\rangle |\bar{0}\rangle + (c + i s) V |\bar{0}\rangle |\bar{1}\rangle \right],
$$

(48)

where $c = \cos(2\tau V)$ and $s = \sin(2\tau V)$. Next note that:

$$
\langle \psi | Z_y(t) = \frac{1}{\sqrt{2}} \left( \langle \phi | \bar{0}\rangle - i \langle \phi | \bar{1}\rangle \right) Z_y(t) = \frac{1}{\sqrt{2}} \left( \langle \phi | \bar{0}\rangle + i \langle \phi | \bar{1}\rangle \right) = \frac{1}{2} \left( \langle \bar{0}| \bar{0}\rangle + i \langle \bar{0}| \bar{1}\rangle - i \langle \bar{1}| \bar{0}\rangle + \langle \bar{1}| \bar{1}\rangle \right).
$$

(49)

Thus:

$$
C(t, r) = \langle Z_y(t) X_x(t) \rangle = \frac{i}{2} \left( (c - i s) V - (c + i s) V \right) \geq \frac{1}{3^{d+2} 2^{d} \alpha} \frac{t^{2d+1}}{\ell^{d+1}},
$$

(50)

where we have used the bound Eq. (40). Therefore, this demonstrates the connected correlators may grow along a superlinear light cone for all $\alpha < 2d + 1$.

We note that in our setting, we only assume the initial state is a bipartite product state across the cut $C$. Our bound therefore also applies to a more restrictive scenario where the initial states are fully product. However, it is not clear whether the bound can be saturated in this scenario.

### 3.4. Simulation of local observables

In this subsection, we use the Lieb-Robinson bounds to improve the performance of quantum algorithms in estimating local observables in time-evolved states. Given an initial state $|\psi\rangle$ and a power-law Hamiltonian $H$, we consider the task of estimating the expectation value of the time-evolved observable $\langle A(t) \rangle := \langle \psi | U(t)^{\dagger} AU(t) | \psi \rangle$ on a quantum computer, where $U(t)$ is the unitary generated by $H$ at time $t$, for a local operator $A$. The ability to perform this task for any arbitrary local observable is equivalent to the ability to compute local density matrices of the time-evolved state $U(t) |\psi\rangle$ or the ability to sample local observables in $U(t) |\psi\rangle$.

A typical approach to estimating $\langle A(t) \rangle$ is as follows. First the unitary evolution $U(t)$ on the entire system is decomposed into a more tractable sequence of elementary unitaries that are supported on a smaller number of sites to produce an approximation to the time-evolved state $|\psi(t)\rangle$. The expectation value $\langle A(t) \rangle$ is then computed by simulating measurements of $A$ on this state. The number of elementary unitaries in the decomposition of $U(t)$ typically increases with both time $t$ and the number of sites $N$ in the system.

However, in the Heisenberg picture, the intuition from the Lieb-Robinson bounds suggests that the dynamics of $A(t) = U(t)^{\dagger} AU(t)$ is mostly confined inside some light cones and, therefore, it should be sufficient to simulate the unitary generated by the Hamiltonian inside the light cones alone. The following result provides such an approximation.
We then use the bounds in Refs. [8, 24] to bound the commutator norm provided, we need to add the cost of preparing $\left\| A(t) - \tilde{A}(t) \right\| = K t^{2d+2} \log \frac{2d r}{\alpha - d}$. \hfill \(51\)

Proof. Without loss of generality, assume that $A$ is initially supported at the origin. Using the triangle inequality, we bound the difference between $A(t)$ and $\tilde{A}(t)$:

$$\left\| A(t) - \tilde{A}(t) \right\| \leq \int_0^t ds \left\| H - H_r, \tilde{A}(s) \right\| \leq \int_0^t ds \sum_{j:D(i,0) > r} \sum_{j:D(i,0) \leq r} \left\| [h_{i,j}, \tilde{A}(s)] \right\|. \hfill \(52\)$$

We then use the bounds in Refs. [8, 24] to bound the commutator norm $\left\| [h_{i,j}, A(s)] \right\|$. For that, we separate the sums over $i$ into terms corresponding to $i$'s inside and outside the linear light cone defined by $D(i, 0) = 2\bar{v}s$.

For $i$ such that $D(i, 0) \leq 2\bar{v}s$, we simply bound $\left\| [h_{i,j}, A(s)] \right\| \leq 2/D(i, j)^\alpha$. Note that in this case, we have $D(i, j) \geq D(i, 0) - 2\bar{v}s \geq D(i, 0)/2$. Therefore, we have

$$\sum_{j:D(i,0) > r} \sum_{j:D(i,0) \leq 2\bar{v}s} \left\| [h_{i,j}, A(s)] \right\| \leq 4^{d+1} \bar{v}^d \sum_{j:D(i,0) > r} \sum_{j:D(i,0) \leq 2\bar{v}s} s^d \frac{D(i, j)^\alpha}{D(i, 0)^\alpha} \leq Ks^d \frac{r^\alpha}{D(i,0)^\alpha}. \hfill \(53\)$$

for some constant $0 < K < \infty$. On the other hand, for $i$ such that $r \geq D(i, 0) > 2\bar{v}s$, we further divide into two cases: $s \geq 1$ and $s < 1$. For $s \geq 1$, we use Ref. [15]:

$$\sum_{j:D(i,0) > r} \sum_{j:D(i,0) > 2\bar{v}s} \left\| [h_{i,j}, A(s)] \right\| \leq K_1 \sum_{j:D(i,0) > r} \sum_{j:D(i,0) > 2\bar{v}s} s^{2d+1} \frac{\log D(i, 0)}{D(i, 0)^\alpha} \leq K_2 s^{2d+1} \sum_{j:D(i,0) > r} \frac{\log D(i, 0)}{D(i, 0)^\alpha} \leq K_3 s^{2d+1} \frac{\log r}{r^\alpha}, \hfill \(54\)$$

where we have used Eq. (16) and defined another set of constants $0 < K_{1,2,3} < \infty$. Similarly, for $s < 1$, we use a bound from Ref. [8] to show that there exists $0 < K_4 < \infty$ such that

$$\sum_{j:D(i,0) > r} \sum_{j:D(i,0) > 2\bar{v}s} \left\| [h_{i,j}, A(s)] \right\| \leq \frac{K_4}{r^\alpha}. \hfill \(55\)$$

Substituting Eq. (53), Eq. (54), and Eq. (55) into Eq. (52) and integrating over time, we obtain Eq. (51). \hfill \(\Box\)

We first consider estimating $\langle A(t) \rangle$ using quantum algorithms. For simplicity, we assume that the Hamiltonian is time-independent in the following discussion. In order for the error of the approximation to be at most a constant, we choose

$$r \propto \max \left\{ t^{\frac{2d+1}{\alpha - d}}, \log t, t \right\}. \hfill \(56\)$$

Therefore, to estimate $\langle A(t) \rangle$, it is sufficient to simulate the evolution of $\tilde{A}(t)$ on $N_r \propto r^d$ sites (instead of simulating the entire lattice).

We then compute $\langle \tilde{A}(t) \rangle$ by simulating $e^{-iH_r t}$ using one of the existing quantum algorithms. Using the $p$th-order product formula for simulating power-law Hamiltonians [25], we need

$$O\left(\left( N_{r,t} \right)^{\alpha_2 + o(1)} \right) = O \left( \frac{t^{(\alpha + d)2 + o(1)}}{\alpha - d} \log t \right), O \left( \frac{t^{(\alpha + d)} + o(1)}{t^{(\alpha + d) + o(1)}} \right) \hfill \(57\)$$

elementary quantum gates, where $o(1)$ denotes $p$-dependent constants that can be made arbitrarily small by increasing the order $p$. For all $\alpha > 2d + 1$, this gate count is less than the estimate without using the Lieb-Robinson bound in Ref. [25]. In particular, in the limit $\alpha \to \infty$, the gate count reduces to $O \left( t^{1 + o(1)} \right)$, which corresponds to the space-time volume inside a linear light cone.

We note that in estimating the gate count for computing $\langle A(t) \rangle$, we have implicitly assumed that we have access to many quantum copies of the initial state $\langle \psi \rangle$. However, in scenarios where only a classical description of $\langle \psi \rangle$ is provided, we need to add the cost of preparing $\langle \psi \rangle$ to the total gate count of the simulation.
4. FROBENIUS LIGHT CONE

We now turn to a stronger notion of light cone, inspired by recent developments in the theory of many-body quantum chaos [19, 20]. Let us consider the early time expansion of a time evolving operator $O_i$, initially supported on lattice site $i$

$$O_i(t) = \sum_{n=0}^{\infty} \frac{(Lt)^n}{n!} O_i = O_i + it[H,O_i] - \frac{t^2}{2}[H,[H,O_i]] + \cdots$$

For illustrative purposes, we have temporarily assumed $H$ is time-independent. Suppose further that $H$ only contains nearest neighbor interactions. Then $[H,O_i]$ can only contain operators of the form $O_{i-1}O_{i+1}$, and $[H,[H,O_i]]$ can contain terms no more complicated than $O_{i-2}O_{i-1}O_{i+1}O_{i+2}$, and so on. It is natural to ask “how much” of the operator can be written as a sum of products of single-site operators restricted to some given subset of the lattice $\Lambda$. This question is naturally interpreted as follows: upon expanding $|O_i(t)\rangle$ in terms of the basis vectors of Eq. (4):

$$|O_i(t)\rangle := \sum_{\{a_k\}} c_{\{a_k\}}(t) \otimes T^a_k,$$

the coefficients $c_{\{a_k\}}(t)$ are analogous to the probability amplitudes of an ordinary quantum mechanical wave function. As we will see, the coefficients $c_{\{a_k\}}(t)$ must be sufficiently small if any $a_k$ are non-identity, when the sites $i$ and $k$ are sufficiently far apart, at any fixed time $t$: this is, intuitively, what we will call the Frobenius light cone.

For mathematical convenience in the discussion that follows, we restrict our analysis to finite lattices. It appears straightforward, if slightly tedious, to generalize to infinite lattices through an appropriate limiting procedure. More significantly, we will focus our discussion to one-dimensional lattices, as only in one dimension have we developed the machinery powerful enough to qualitatively improve upon the results of Section 3.

4.1. A vector space of operators

We define a one dimensional lattice

$$\Lambda := \{i \in \mathbb{Z} : 0 \leq i \leq L\}.$$ (60)

For every site $i \in \Lambda$, we assume a finite dimensional local Hilbert space $\mathcal{H}_i$, obeying $\dim(\mathcal{H}_i) < \infty$. The global Hilbert space is

$$\mathcal{H} := \bigotimes_{i \in \Lambda} \mathcal{H}_i.$$ (61)

Let $\mathcal{B}$ denote the set of Hermitian operators acting on $\mathcal{H}$. We equip this space with the Frobenius inner product

$$(A|B) := \frac{\text{tr}(AB)}{\dim(\mathcal{H})},$$ (62)

upon which $\mathcal{B}$ becomes a real inner product space; we denote elements of this vector space $O \in \mathcal{B}$ as $|O\rangle$. When $A = B$, the inner product reduces to the squared Frobenius norm of $A$: $(A|A) = \|A\|^2_F$. Note that for traceless operators $A$ and $B$, this inner product corresponds to the value of the thermal two-point connected correlation function at infinite temperature. Let $\{T^a_i\}$ denote the generators of $U(\dim(\mathcal{H}_i))$, with $a = 0$ denoting the identity matrix. These generators form a complete basis for $\mathcal{B}$:

$$\mathcal{B} := \text{span} \left\{ \bigotimes_{i \in \Lambda} T^a_i \right\} := \text{span} \{ |a_0 \cdots a_L\rangle \}.$$ (63)

We define the projectors

$$Q_x[a_0 \cdots a_L] := \left\{ \begin{array}{cl} |a_0 \cdots a_L\rangle & x \neq 0 \text{ and } a_y = 0 \text{ if } y > x \\ 0 & \text{otherwise} \end{array} \right.$$ (64)
Our goal is to show that $B^\alpha$ with power-law interactions of exponent $\alpha$ are orthogonal and complete:

\[
Q_i Q_j = \delta_{ij} Q_j, \quad \sum_{i \in \Lambda} Q_i = 1. \tag{65}
\]

Time evolution is generated by a (generally time-dependent) Hamiltonian $H(t) : \mathbb{R} \to \mathcal{B}$. We assume that $H$ is 2-local:

\[
H(t) = \sum_{(i,j) \in \Lambda} H_{ij}(t), \tag{66}
\]

with power-law interactions of exponent $\alpha$. By unitarity,

\[
(\mathcal{O}|\mathcal{L}(t)|\mathcal{O}) = 0, \tag{67}
\]

where $\mathcal{L}(t)$ was defined in Eq. (12); hence $\mathcal{L}(t)$ generates orthogonal transformations on $\mathcal{B}$ and leaves the length of all operators invariant.

### 4.2. The operator quantum walk

Our goal is to understand the following scenario (Fig. 4): given an operator $|\mathcal{O}\rangle$ starting at the left-most site, i.e. obeying $Q_0 O = |\mathcal{O}\rangle$, how long does it take before most of the operator $|\mathcal{O}(t)\rangle$ consists of operator strings that act on sites $\geq x$? More precisely, define

\[
t_2^\alpha(x) := \inf \left\{ t > 0 : \text{for any } Q_0|\mathcal{O}_0\rangle = |\mathcal{O}_0\rangle, \quad \delta > \frac{\sum_{y:x \leq y \leq L} (|\mathcal{O}_0(t)||q_y||\mathcal{O}_0(t))}{(|\mathcal{O}_0||\mathcal{O}_0|)} \right\} \tag{68}
\]

to be the shortest time for which a fraction $\delta$ of the operator $|\mathcal{O}(t)\rangle$ cannot be supported on sites $< x$. The assumption that the operator starts only on the left-most site is not restrictive—for an initial site $k \in \Lambda$, we can identify the lattice sites $k + m \sim k - m$ in order to “fold” the one dimensional lattice to put the initial point $k$ at one boundary; such a change cannot modify Eq. (14), except to adjust the value of $\hbar$ by a factor $< 4$.

We note that

\[
\sup_{\mathcal{O}_x \in \mathcal{B}_x} \frac{\text{tr} \left( |\mathcal{O}_0(t), \mathcal{O}_x\rangle \langle \mathcal{O}_0(t), \mathcal{O}_x\rangle \right)}{\dim(\mathcal{H}) (|\mathcal{O}_0||\mathcal{O}_0|)} \leq 4 \frac{(|\mathcal{O}_0(t)||q_x||\mathcal{O}_0(t))}{(|\mathcal{O}_0||\mathcal{O}_0|)} \leq 4 \sum_{y:x \leq y \leq L} (|\mathcal{O}_0(t)||q_y||\mathcal{O}_0(t)) \tag{69}
\]

where the left-most side corresponds to the out-of-time-order correlation function (OTOC) of an infinite-temperature state—a quantity known to herald the onset of many-body quantum chaos [19, 20]. From Eq. (69), it follows that a lower bound on $t_2^\alpha(x)$ also bounds the evolution time of the OTOC and the growth of chaos.

The second main result of this paper is the following theorem:

**Theorem 7.** Given Hamiltonian evolution on $\mathcal{H}$ obeying Eq. (66) and Eq. (14), for any $x \in \Lambda$, $0 < \delta \in \mathbb{R}$ and $\frac{3}{2} < \alpha \in \mathbb{R}$, there exist constants $0 < K, K' < \infty$ such that

\[
t_2^\alpha(x) \geq K \times \begin{cases} 
\frac{x}{x^{\alpha-3/2}(1+K'\log x)}^{-1} & \alpha > \frac{5}{2} \\
\frac{\alpha}{2} & \frac{3}{2} < \alpha \leq \frac{5}{2}
\end{cases}, \tag{70}
\]

**Proof.** We prove this theorem using the “operator quantum walk” formalism introduced in Ref. [26]. For simplicity, we will first prove the theorem when $\alpha > \frac{5}{2}$, and then generalize to $\alpha \leq \frac{5}{2}$ afterwards. Consider the operator $\mathcal{F}$ acting on $\mathcal{B}$ defined by

\[
\mathcal{F} := \sum_{j \in \Lambda} j Q_j. \tag{71}
\]

Our goal is to show that

\[
\lim_{L \to \infty} \| [\mathcal{F}, \mathcal{L}(t)] \|_{\infty} \leq C < \infty. \tag{72}
\]
The 4-dimensional space of operators can be decomposed into direct sum of $L$ subspaces $\{Q_i\}$ by the position of the right-most occupied site. By keeping track of only the “average value” of the right-most site (depicted above), keeping in mind that an exponential number of orthogonal operators (depicted below) are contained on most of the sites, we reduce the quantum walk of many-body operators from an exponentially large space to a one dimensional line.

The reason Eq. (72) is desirable is the following. Without loss of generality, we normalize $\langle O|O \rangle = 1$. We then define a time-dependent probability distribution $P_t$ on $\Lambda$ as

$$P_t(i \in \Lambda) := \langle O(t)|Q_i|O(t) \rangle,$$

since by Eq. (65) the probability distribution is properly normalized: $P_t(\Lambda) = 1$. We may then reinterpret $t_2^\alpha(x)$ as the first time where the probability that $i \geq x$ on the measure $P_t$ is sufficiently large:

$$t_2^\alpha(x) = \inf \{ t > 0 : \delta > P_t(i \geq x) \}.$$

We may then interpret $\mathcal{F}$ for $\alpha > \frac{5}{2}$ as a classical random variable that gives $i$ with probability $P_t(i)$. By Markov’s inequality,

$$P_t(i \geq x) \leq \frac{E_t[\mathcal{F}]}{x},$$

where $E_t[\cdot]$ denotes expectation value on the measure $P_t$. If Eq. (72) holds, then for any operator $O_0$ in the domain of $Q_0$,

$$E_t[\mathcal{F}] = \int_0^t ds \frac{d}{ds} \langle O_0(s)|\mathcal{F}|O_0(s) \rangle = \int_0^t ds \langle O_0(s)|[\mathcal{F}, L(s)]|O_0(s) \rangle$$

$$\leq \int_0^t ds |\langle O_0(s)|[\mathcal{F}, L(s)]|O_0(s) \rangle| \leq C t.$$

Combining Eq. (75) and Eq. (76), we see that Eq. (70) holds with

$$K = \frac{\delta}{C}.$$
Hence, it remains to prove Eq. (72). To do so, it will be useful to define
\[ \overline{\Lambda} := \Lambda - \{0\} , \] (78)
and a more refined set of complete, orthogonal projectors: for \( S \subseteq \overline{\Lambda} \),
\[ \mathbb{R}_S | a_0 \cdots a_L \rangle := \begin{cases} |a_0 \cdots a_L \rangle & \text{if } i > 0 \text{ and } a_i \neq 0 \text{ if and only if } i \in S \\ 0 & \text{otherwise} \end{cases} \] (79)
which projects onto the operators whose support is exactly the subset \( S \). We also define
\[ F_S := \max_{i \in S} i \] (80)
to be the right-most occupied site. Observe that \( F_S \mathbb{R}_S = \mathbb{R}_S F \mathbb{R}_S \). Since
\[ \sum_{S \in \mathbb{Z}_2^\Lambda} \mathbb{R}_S = 1 , \] (81)
we may write, for any \( O \in \mathcal{B} \),
\[ (O| [F, L]| O) = \sum_{S, Q \in \mathbb{Z}_2^\Lambda} (O| \mathbb{R}_S [F, L] \mathbb{R}_Q| O) \leq \sum_{S, Q \in \mathbb{Z}_2^\Lambda} \sqrt{(O| \mathbb{R}_S| O)(O| \mathbb{R}_Q| O)} \sup_{\mathcal{O}, \mathcal{O}' \in \mathcal{B}} \frac{(F_S - F_Q)(O| \mathbb{R}_S \mathcal{L} \mathbb{R}_Q| \mathcal{O}')}{\sqrt{(O| O)(O'| O')}} . \] (82)
Next, we observe that the 2-locality of the Hamiltonian implies that \( \mathbb{R}_S \mathcal{L} \mathbb{R}_Q \neq 0 \) if and only if there exists a site \( i \in \Lambda \) such that \( S \cup \{i\} = Q \) or \( Q \cup \{i\} = S \).
Suppose that \( Q \cup \{i\} = S \), that \( F_Q = j \) and that \( i > 0 \). Then if \( i < j \), \( F_S = F_Q = j \); the right-most occupied site in \( S \) and \( Q \) has not changed, and hence the supremum in Eq. (82) vanishes. Therefore, the supremum is only non-trivial when \( i > j \). By submultiplicativity of the operator norm, there exists \( 0 < A < \infty \) such that
\[ \sup_{\mathcal{O}, \mathcal{O}' \in \mathcal{B}} \left| \frac{(F_S - F_Q)(O| \mathbb{R}_S \mathcal{L} \mathbb{R}_Q| \mathcal{O}')}{\sqrt{(O| O)(O'| O')}} \right| \leq 2|i - j| \left\| \sum_{k \in Q} H_{k i} \right\|_\infty \leq 2|i - j| \sum_{k \in Q} \frac{h}{|i - k|^\alpha} \leq \frac{A}{|i - F_Q|^\alpha - 2} , \] (83)
where \( A \) is a constant and, in the last step, we overestimated the sum by assuming all sites \( \leq j \) are included in the set \( Q \). A similar argument holds when \( S \cup \{i\} = Q \).

It is now useful to interpret Eq. (82) as an auxiliary linear algebra problem. Let us define \( \varphi_S \in \mathbb{R}^{\mathbb{Z}_2^\Lambda} \) as
\[ \varphi_S := \sqrt{(O| \mathbb{R}_S| O)} , \] (84)
and \( M \in \mathbb{R}^{\mathbb{Z}_2^\Lambda \times \mathbb{Z}_2^\Lambda} \) as
\[ M_{SQ} = M_{QS} := \begin{cases} A|F_S - F_Q|^{2-\alpha} & F_S \neq F_Q \text{ and } S = Q \cup \{m\} \text{ or } Q = S \cup \{m\} \\ 0 & \text{otherwise} \end{cases} . \] (85)
Since
\[ (O| [F, L]| O) \leq \sum_{S, Q} \varphi_S M_{SQ} \varphi_Q = \left\| M \right\|_\infty , \] (86)
it suffices to show that \( \left\| M \right\|_\infty < \infty \).
To bound the maximal eigenvalue of \( M \), we use the min-max Collatz-Weiland Theorem [27]. To do that, we must first establish that \( M \) is an irreducible matrix (non-negativity of the entries is guaranteed by Eq. (85)). To show irreducibility, we observe that
\[ \left( M^{[S]} \right)_{S} \neq 0 ; \] (87)
the sequence of subsets which satisfies this identity corresponds to sequentially adding the elements of \( S \) from smallest to largest. We conclude that (by non-negativity of all \( M^n \)) there exists an \( n \in \mathbb{Z}_+^+ \) such that \( (M^n)_{SQ} > 0 \) for all sets \( S \) and \( Q \).
We are now ready to apply the min-max Collatz-Weiland Theorem:

\[
\|M\|_\infty = \inf_{\varphi \in \mathbb{R}^Z \setminus \{0\}} \max_S \frac{1}{\varphi_S} \sum_{Q \in \mathbb{Z}^\Lambda} M_{SQ} \varphi_Q. \tag{88}
\]

Clearly an upper bound to the maximal eigenvalue comes from choosing any trial vector \(\varphi\) that we desire. We make the following choice: writing

\[
S = \{n_1, \ldots, n_\ell\}, \text{ with } n_i < n_{i+1},
\]

we take \(\varphi_\emptyset = 1\), and then define \(n_0 = 0\) and

\[
\varphi_S := \prod_{i=1}^{\lfloor |S| \rfloor} (n_i - n_{i-1})^{-\beta}, \tag{90}
\]

where \(\beta\) is a tunable parameter we will shortly fix. Now we evaluate the right hand side of Eq. (88), defining \(j = F_S:\n\]

\[
\frac{1}{\varphi_S} \sum_{Q \in \mathbb{Z}^\Lambda} M_{SQ} \varphi_Q = M_{S,S-(j)} \frac{\varphi_{S-(j)}}{\varphi_S} + \sum_{k \in \Lambda : k > j} M_{S,S \cup \{k\}} \frac{\varphi_{S \cup \{k\}}}{\varphi_S} \tag{91}
\]

Using Eq. (85), and assuming that \(j_* = F_{S-(j)}:\n\]

\[
M_{S,S-(j)} \frac{\varphi_{S-(j)}}{\varphi_S} \leq \alpha(j - j_*)^{2-\alpha}. \tag{92}
\]

We hence take

\[
\beta = \alpha - 2 \tag{93}
\]

to ensure that this first term is finite. Evaluating the second term of Eq. (91),

\[
\sum_{k \in \Lambda : k > j} M_{S,S \cup \{k\}} \frac{\varphi_{S \cup \{k\}}}{\varphi_S} \leq A \sum_{k=j+1}^{\infty} (k - j)^{2-\alpha - \beta} \leq A_*, \tag{94}
\]

where

\[
A_* := A \frac{2^\alpha + \beta - 3}{\alpha + \beta - 3} < \infty, \tag{95}
\]

so long as \(\alpha > \frac{5}{2}\). We conclude that \(C \leq A + A_* < \infty\), proving the theorem when \(\alpha > \frac{5}{2}\).

We now return to the case \(\frac{5}{2} < \alpha \leq \frac{5}{2}\). The proof is essentially identical with a few minor changes. Firstly, we set \(F_{(0)} = 0\), and for non-empty sets we define

\[
F_S := \max_{j \in S} \frac{j^\gamma}{1 + K' \log j}, \tag{96}
\]

for a parameter \(\gamma \in (0, 1)\) that we will fix shortly. We choose the parameter \(K'\) such that \(F_i\) is a convex function on \(\mathbb{Z}^+\): \(|F_i - F_j| \leq F_{|i-j|}\). Such a \(K'\) can be shown to exist by extending \(F\) to act on \([1, \infty)\), after which we use elementary calculus to demand that

\[
0 < \frac{dF(x)}{dx} = \frac{1}{x^{1-\gamma}(1 + K' \log x)} \left(\gamma - \frac{K'}{1 + K' \log x}\right), \tag{97}
\]

along with

\[
0 > \frac{d^2F(x)}{dx^2} = -\frac{1}{x^{2-\gamma}(1 + K' \log x)} \left(1 - \gamma + \frac{K'}{1 + K' \log x}\right) \left(\gamma - \frac{K'}{1 + K' \log x}\right) - \left(\frac{K'}{1 + K' \log x}\right)^2. \tag{98}
\]
Eq. (97) and Eq. (98) are both satisfied by the choice
\[ K' = \frac{\gamma}{4}. \] (99)

We then find that convexity of \( F_i \) leads to the replacement of Eq. (85) with
\[ M_{SQ} = M_{QS} := \left\{ \begin{array}{ll} A|F_S - F_Q|^\gamma (1 + K' \log |F_S - F_Q|)^{-1} & F_S \neq F_Q \\ 0 & \text{otherwise} \end{array} \right. . \] (100)

Lastly, we replace Eq. (90) with
\[ \varphi_S := \prod_{i=1}^{[S]} \frac{(n_i - n_{i-1})^{\gamma+1-\alpha}}{1 + K' \log(n_i - n_{i-1})}. \] (101)

These choices guarantee that
\[ M_{S,S\setminus\{j\}} \frac{\varphi_{S\setminus\{j\}}}{\varphi_S} = A, \] (102)
as in the prior setting. Then we find that
\[ \sum_{k \in \Lambda : k>j} M_{S,S \cup \{k\}} \frac{\varphi_{S \cup \{k\}}}{\varphi_S} \leq A \sum_{k=j+1}^{\infty} \frac{1}{(k-j)^{2(\alpha-1-\gamma)}(1 + K' \log(k-j))^2}. \] (103)

Upon choosing \( \gamma = \alpha - \frac{3}{2} \), we obtain that the sum above is finite. Note that the logarithmic factors were required to obtain finiteness of Eq. (103). Hence we obtain \( \|M\|_\infty < \infty \). Lastly, we mimic the arguments of Eq. (76) to complete the proof. \( \square \)

We conjecture that in \( d > 1 \), the Frobenius light cone is always linear if and only if
\[ \alpha > \frac{3d}{2} + 1. \] (104)

We expect that for \( q \)-local Hamiltonians with \( q > 2 \), Eq. (104) holds only when a slightly stricter requirement than Eq. (14) is obeyed: for example, if \( \|H_{\{n_1, \ldots, n_q\}}\| \lesssim \prod_i |n_i - n_{i+1}|^{-\alpha} \) in one dimension.

The Frobenius light cone of Theorem 7 is tight up to subalgebraic corrections, when applied to arbitrarily large operators. This can be seen by considering a large operator of the form
\[ O_0 = \prod_{i=0}^{L/2} X_i^+ + \text{h.c.} \] (105)
where \( X_i^+ = X_i + iY_i \). If the Hamiltonian is
\[ H = \sum_{0 \leq j \leq L/2} \sum_{L/2 < k \leq L} \frac{Z_j Z_k}{L^\alpha}, \] (106)
it is straightforward to show that the fraction of \( O_0(t) \) supported beyond \( L/2 \) is (up to the first order in \( t \))
\[ \sum_{k > L/2} Q_k O_0(t) = \frac{t}{2L^\alpha-1} O_0 \sum_{L/2 < k \leq L} Z_k. \] (107)
The Frobenius norm of this fraction is
\[ \left( \frac{L}{2} \left( \frac{t}{2L^\alpha-1} \right)^2 \right)^{\frac{1}{2}} \propto \frac{t}{L^{\alpha-3/2}}. \] (108)

Therefore, our bound in Theorem 7 is tight up to \( O(1) \) factors.
4.3. Quantum state transfer

An immediate consequence of this theorem is that the Lieb-Robinson light cone is not relevant for infinite temperature many-body quantum chaos and the growth of operators. A more practical application of the Frobenius light cone are tighter constraints on quantum state transfer. For simplicity, we assume that dim(H_i) = 2, and denote |0_i⟩ and |1_i⟩ as the eigenstates of the Pauli matrix Z_i on H_i.

A strong notion of quantum state transfer from i ∈ Λ to j ∈ Λ, which is independent of the background state, is to demand that there exists a Hamiltonian protocol H(t) and a time τ ∈ R such that

\[ X^o_i(τ) = X^o_j. \] (109)

It is obvious that Theorem 7 constrains the time at which Eq. (109) may hold; hence Eq. (109) cannot be performed at a time τ which scales slower than linearly in the distance D(i, j) when α > 5/2.

Alternatively, we may consider the following definition of weak state transfer from i to j. Consider a quantum state whose initial condition is

\[ |ψ(0)⟩ := |φ_i⟩ ⊗ \bigotimes_{k ∈ Λ \setminus \{i\}} |0_k⟩. \] (110)

for arbitrary |φ_i⟩ ∈ H_i. Our goal is to find a time evolution operator U(t) and a time τ, such that |ψ(t)⟩ = U(t)|ψ(0)⟩ and

\[ ⟨ψ(τ)|Z_j|ψ(τ)⟩ = ⟨φ_i|Z_j|φ_i⟩. \] (111)

In particular, the probability of measuring a 0 or 1 on site j at time τ is given by the probability of measuring it at time t = 0 on site i. This property must hold for all |φ_i⟩ for a fixed U(t). We consider a time evolution operator U(t) that obeys

\[ U(t) \bigotimes_{k ∈ Λ} |0_k⟩ = \bigotimes_{k ∈ Λ} |0_k⟩, \] (112)

and which is generated by an arbitrary long-range Hamiltonian Eq. (66) of exponent α. This definition includes more conventional definitions of state transfer, and is weaker as it ignores the relative phase between the two states of |φ_i⟩. It suffices to constrain this weak notion of state transfer to constrain stronger state-transfer protocols that transfer all the quantum information encoded in a given state.

**Corollary 8.** Let 3/2 < α ∈ R and x = D(i, j). In the weak state-transfer protocol above, there exist \(0 < K, K' < ∞\) such that

\[ τ > K × \left\{ \begin{array}{ll} x & \text{if } x < 3/2 \log x^{-1} \frac{α}{3/2} \leq \frac{5}{2} \leq \frac{5}{2} \alpha > \frac{5}{2} \end{array} \right. \] (113)

**Proof.** We begin by observing that we may assume |φ_i⟩ = |1_i⟩ without loss of generality, since Eq. (111) is trivially obeyed by Eq. (112). Now the proof largely mirrors that of Theorem 7. Without loss of generality, we may define lattice sites such that i = 0 and j > 0, as explained above. Define

\[ |S⟩ := \bigotimes_{k ∈ S} |1_k⟩ \otimes \bigotimes_{k ∈ S^c} |0_k⟩, \] (114)

and the observable F which acts on the mutual eigenbasis of Z_i as

\[ F|S⟩ := F_S|S⟩, \] (115)

for any \(S ⊆ Λ\); here F_S is given by Eq. (80) when \(α > 3/2\) and Eq. (96) when \(3/2 < α \leq 5/2\). For simplicity we only describe explicitly the case \(α > 3/2\), as the other case follows from identical considerations. We evaluate

\[ \frac{d}{dt}⟨ψ(t)|F|ψ(t)⟩ \leq -i⟨ψ(t)|[F, H(t)]|ψ(t)⟩ \leq ∥F, H(t)∥_∞. \] (116)

As before, our goal is to show that \(∥F, H(t)∥_∞ < ∞\). Since H is 2-local, we know that \(H_{ij}(t)|0_i⟩⟨0_j| \propto |0_i⟩|0_j⟩\) by Eq. (112). This implies that, as before \([F, H]\) can only be non-vanishing when H serves to either add a new \(|1⟩\)}
to the right end of the state, or delete the right-most $|1\rangle$. Hence $(S|H(t)|Q) \neq 0$ only if $|S - S \cap Q| \leq 1$ and $|Q - S \cap Q| \leq 1$. We define the matrix $MSQ := \sup(S|F,H(t)|Q)$, which equals

$$MSQ = M_{QS} := \begin{cases} 
A|F_S - F_Q|^2 - \alpha & S = Q \cup \{m\} \text{ or } Q = S \cup \{m\} \\
A|F_S - F_Q|^1 - \alpha & \text{there exists } R \text{ with } S = R \cup \{m\} \text{ and } Q = R \cup \{n\}, \text{ and } Q \neq S \text{ and } F_R < \min(F_S,F_Q), \\
0 & \text{otherwise}
\end{cases} \tag{117}
$$

We bound the maximal eigenvalue of $M$ using the Collatz-Wieland inequality Eq. (88), using the trial vector $\varphi_S$ given Eq. (90). Observe that the first line of Eq. (117) is identical to Eq. (85); as such these terms in $M_{SQ}\varphi_Q$ are bounded by a constant as before. The new terms we must deal with arise from the second line of Eq. (117). If $S$ is given by Eq. (89), we find that

$$\frac{1}{\varphi_S} \sum_{Q:|Q|=|S|} M_{SQ}\varphi_Q < A \sum_{m=n_{\ell-1}+1}^{\infty} \left(\frac{n_{\ell} - n_{\ell-1}}{m - n_{\ell-1}}\right)^{a-2} \frac{1 - \delta_{m,n_\ell}}{(m - n_{\ell-1})^{a-1}} < A_{st}, \tag{118}
$$

for some constant $0 < A_{st} < \infty$, so long as $\alpha > \frac{5}{2}$. We conclude that $M$ has a bounded maximal eigenvalue, independently of the lattice size. We conclude there exists $0 < K < \infty$ such that $(\psi(t)|F|\psi(t)) \leq Kt$.

At time $\tau$, we must have

$$|\psi(\tau)\rangle = |1_j\rangle \otimes |\psi'_{A -(j)}\rangle, \tag{119}
$$

for arbitrary state $|\psi'\rangle$ acting on sites other than $j$. Therefore,

$$(\psi(\tau)|F|\psi(\tau)) \geq j. \tag{120}
$$

Using Markov’s inequality as in the proof of Theorem 7, we obtain Eq. (113). The case $\alpha < \frac{5}{2}$ is proved analogously. \hfill $\square$

## 5. FREE LIGHT CONE

In this section, we discuss bounds on the quantum dynamics of non-interacting many-body systems.

### 5.1. Non-interacting Hamiltonians

Consider a many-body quantum system defined on a $d$-dimensional lattice graph $\Lambda$: we assume the same properties of $\Lambda$ as in Section 2. Suppose that the many-body Hamiltonian takes the form

$$H(t) = \sum_{i,j \in \Lambda} h_{ij}(t)c_i^\dagger c_j, \tag{121}
$$

where $h(t) : \mathbb{R} \to \mathbb{C}^{\Lambda \times \Lambda}$ is a Hermitian matrix, and $c_i^\dagger$ and $c_i$ represent either fermionic creation and annihilation operators:

$$\{c_j, c_i^\dagger\} := \delta_{ij}, \tag{122}
$$

or bosonic creation and annihilation operators:

$$[c_j, c_i^\dagger] := \delta_{ij}. \tag{123}
$$

The on site Hilbert space $\mathcal{H}_i$ obeys dim($\mathcal{H}_i$) = 2 in the fermionic case, and dim($\mathcal{H}_i$) = $\infty$ in the bosonic case. We note, however, that in isolated bosonic systems, $\mathcal{H}_i$ can often be truncated so that dim($\mathcal{H}_i$) is at most the number of excitations on the lattice and is therefore finite.

As is well known, the evolution of all operators in such a non-interacting theory is controlled by the Green’s function of the single particle problem on the Hilbert space $\mathbb{C}^\Lambda$. Time evolution on this space is generated by the Hamiltonian

$$H_{sp}(t) := \sum_{i,j \in \Lambda} h_{ij}(t)|i\rangle\langle j|. \tag{124}$$
The single particle time evolution matrix obeys the differential equation

$$\frac{d}{dt} U_{sp}(t) := -i H_{sp}(t) U_{sp}(t),$$

(125)

together with the initial condition $U_{sp}(0) = 1$. For example, in the fermionic model,

$$c_i(t) = \sum_{j \in \Lambda} U_{sp,ij}(t)c_j,$$

(126)

which follows from observing that

$$\frac{d}{dt} c_i := i [H(t), c_i] = \sum_{j \in \Lambda} i h_{ij}(t) [c^*_i c_j, c_i] = -i \sum_{j \in \Lambda} h_{ij}(t) c_j.$$

(127)

For simplicity in the discussion that follows, we drop the “sp” subscript on $H$ and $U$.

5.2. Quantum walks of a single particle

Consider a normalized wave function $|\psi(t)\rangle := U(t)|\psi\rangle \in \mathbb{C}^\Lambda$, along with its canonical probability distribution $P_t$ on $\Lambda$:

$$P_t(A) := \sum_{i \in A} \frac{|\langle i|\psi(t)\rangle|^2}{\langle \psi|\psi \rangle},$$

(128)

Let us label an origin $0 \in \Lambda$, and assume that $|\psi(0)\rangle = |0\rangle$. We now use the quantum walk framework to prove our third main result, on the concentration of $P_t$ on lattice sites close to the origin.

**Theorem 9.** If $\alpha > d + 1$, $\epsilon > 0$, and $r \in \mathbb{Z}^+$, there exist constants $0 < K, u < \infty$ such that

$$\sum_{y \in \Lambda: D(y,0) \geq r} P_t(y) \leq \frac{Kt}{(r -ut)^{\alpha-d-\epsilon}}.$$

(129)

When $d < \alpha \leq d + 1$, Eq. (129) holds with $u = 0$.

**Proof.** We first prove Eq. (129) when $\alpha > d + 1$. Define the Hermitian operator

$$\langle x|\mathcal{F}(t)|y \rangle := \delta_{xy} \mathcal{F}(x,t),$$

(130a)

$$\mathcal{F}(x,t) := \min (0, D(x,0) - ut).$$

(130b)

Our goal is to follow the proof of Theorem 7: first bounding the rate of change of an expectation value, and then employing Markov’s inequality. The operator whose expectation value we will bound in the time evolved wave function is $\mathcal{F}^\beta$; ultimately we will see that $\beta = \alpha - d - \epsilon$.

First, let us bound

$$|\mathcal{F}(x)^\beta - \mathcal{F}(y)^\beta| \leq \beta \max(\mathcal{F}(x), \mathcal{F}(y))^{\beta-1} |\mathcal{F}(y) - \mathcal{F}(x)| \leq \beta D(x,y) \left( \mathcal{F}(x)^{\beta-1} + \mathcal{F}(y)^{\beta-1} \right).$$

(131)

Then,

$$\frac{d}{dt} \langle \psi(t)|\mathcal{F}^\beta|\psi(t)\rangle = -i \langle \psi(t)|[\mathcal{F}^\beta, H(t)]|\psi(t)\rangle - \epsilon \beta \langle \psi(t)|\mathcal{F}^{\beta-1}|\psi(t)\rangle.$$

(132)

Let us first bound the first term, using Eq. (131) and Eq. (132):

$$|\langle \psi(t)|[\mathcal{F}^\beta, H(t)]|\psi(t)\rangle| \leq 2 \sum_{\{x,y\} \subset \Lambda} |\langle x|[\mathcal{F}^\beta, H(t)]|y \rangle| \langle x|\psi\rangle \langle y|\psi \rangle | \leq \sum_{x \in \Lambda} \sum_{y \in \Lambda \setminus \{x\}} (P_t(x) + P_t(y))|\langle x|[\mathcal{F}^\beta, H(t)]|y \rangle|.$$
\[
\leq \beta \sum_{x \in \Lambda} \sum_{y \in \Lambda - \{x\}} \mathbb{P}_t(x) \frac{2h}{D(x,y)^{\alpha-1}} \left( F(x)^{\beta-1} + F(y)^{\beta-1} \right). 
\]  

(133)

In the last line, we have used the symmetry of the sum under exchanging \(x\) and \(y\) to remove \(\mathbb{P}_t(y)\). Then we observe that

\[
F(y)^{\beta-1} \leq (F(x) + D(x,y))^{\beta-1} \leq 2^{\beta-1} (F(x)^{\beta-1} + D(x,y)^{\beta-1}) .
\]

(134)

Hence, so long as we choose

\[
\beta = \alpha - d - \epsilon ,
\]

(135)

we conclude that there exist constants \(0 < K, A < \infty\) such that

\[
|\psi(t)||F^\beta, H(t)||\psi(t)| \leq (2 + 2^\beta) \sum_{x \in \Lambda} \sum_{y \in \Lambda - \{x\}} \left( \frac{h}{D(x,y)^{\alpha-\beta}} + \frac{h}{D(x,y)^{\alpha-1}} F(x)^{\beta-1} \right)
\]

\[
\leq \sum_{x \in \Lambda} \mathbb{P}_t(x) \left( K + A F(x)^{\beta-1} \right) = K + A \langle \psi(t) | F^{\beta-1} | \psi(t) \rangle ,
\]

(136)

where \(K, A\) are constants. Upon choosing

\[
u = \frac{A}{\beta} ,
\]

(137)

Eq. (132) implies that

\[
\langle \psi(t) | F^\beta | \psi(t) \rangle \leq Kt.
\]

(138)

Using Markov’s inequality, and assuming \(r > ut\),

\[
\sum_{y \in \Lambda : D(y,x) \geq r} \mathbb{P}_t(y) \leq \frac{\mathbb{E}_t[F^\beta]}{(r - ut)^\beta} \leq \frac{Kt}{(r - ut)^\beta} .
\]

(139)

Combining Eq. (135) and Eq. (139) we obtain Eq. (129).

Secondly, we study the case \(\alpha \leq d + 1\). Now we define

\[
\langle x, F \rangle := \delta_{xy} D(x,0)^\beta ,
\]

(140)

with \(\beta\) given by Eq. (135). Observe that \(\beta < 1\). In this limit,

\[
\frac{d}{dt} \mathbb{E}_t[F] \leq \sum_{x \in \Lambda} \mathbb{P}_t(x) \sum_{y \in \Lambda - \{x\}} \frac{h}{D(x,y)^{\alpha}} |D(x,0)^\beta - D(y,0)^\beta| \leq \sum_{x \in \Lambda} \mathbb{P}_t(x) \sum_{y \in \Lambda - \{x\}} \frac{h}{D(x,y)^{\alpha-\beta}} ,
\]

(141)

where in the last inequality we have used the convexity of \(F\) as a function of distance. For any \(\epsilon > 0\), the sum over \(y\) converges; hence there exists a \(K < \infty\) such that

\[
\frac{d}{dt} \mathbb{E}_t[F] \leq \sum_{x \in \Lambda} \mathbb{P}_t(x) \times K = K .
\]

(142)

Another application of Markov’s inequality implies Eq. (129).

5.3. Local simulation of a single particle

An immediate application of Theorem 9 is to bound the error made by approximating time evolution via a truncated, local Hamiltonian, analogous to the discussion of Section 3.4.
Corollary 10. For any \( i \in \Lambda \), define \( B^r_i := \{ j \in \Lambda : D(j, i) \leq r \} \), and define \( \tilde{H}(t) \) to be the restriction of a free bosonic Hamiltonian \( H(t) \) [Eq. (121)] to \( B^r_i \subset \Lambda \). Then for any \( \epsilon > 0 \), there exists \( 0 < K, K' < \infty \) such that for times
\[
t < \frac{K'}{n^{\min(1, \alpha-d-\epsilon)/3}},
\] we have
\[
\left\| \hat{b}^i_0(t) - \hat{b}^i_0(t) \right\| \leq Kn^{3/2} \left( \frac{t}{r^{\alpha-d}} + \frac{t^{3/2}}{r^{(\alpha-d-\epsilon)/2}} \right),
\]
where the norm is estimated in the subspace that has at most \( n \geq 1 \) excitations across the lattice and \( \tilde{b}^i_0(t) \) denotes time evolution with the restricted Hamiltonian \( \tilde{H}(t) \).

Proof. Without loss of generality, we assume \( i = 0 \), the origin. Observe that
\[
\left\| \hat{b}^i_0(t) - \hat{b}^i_0(t) \right\| \leq \int_0^t ds \left\| \hat{b}^i_0(t), H(t) - \tilde{H}(t) \right\| \leq \int_0^t ds \left\| \hat{b}^i_0(t), \sum_{i: D(i,0) \leq r} \sum_{j: D(j,0) > r} h_{ij} b^i_j \right\|.
\]
Using Theorem 9,
\[
\hat{b}^i_0(t) = \sum_{i: D(i,0) \leq r} f_i(t) b^i_i,
\]
where the coefficients \( f_i(t) \) satisfy, for some \( 0 < C < \infty \) and arbitrary \( \epsilon > 0 \),
\[
\sum_{i: D(i,0) \geq x} |f_i(t)|^2 \leq \frac{Ct}{x^{\alpha-d-\epsilon}},
\]
for all \( x > 0 \) and all \( t \) obeying Eq. (143).

We separate the sum over \( i \) in Eq. (145) according to \( D(i,0) \leq r/2 \) and \( r/2 < D(i,0) \leq r \). In the former case, we have
\[
\left\| \hat{b}^i_0(t), \sum_{i: D(i,0) \leq r/2} \sum_{j: D(j,0) > r} h_{ij} b^i_j \right\| \leq 2 \sqrt{n} \sum_{i: D(i,0) \leq r/2} \sum_{j: D(j,0) > r} h_{ij} b^i_j \leq 2n^{3/2} \max_{i: D(i,0) \leq r/2} \frac{1}{D(i,j)\alpha} \leq \frac{C_1 n^{3/2}}{r^{\alpha-d}},
\]
where \( 0 < C_1 < \infty \) is a constant. We have used the fact that \( |h_{ij}| \leq 1/D(i,j)^\alpha \) and that \( D(j,i) \geq r/2 \) for all \( i \) such that \( D(i,0) \leq r/2 \).

On the other hand, for \( r/2 < D(i,0) \leq r \),
\[
\left\| \hat{b}^i_0(t), \sum_{i: r/2 < D(i,0) \leq r} \sum_{j: D(j,0) > r} h_{ij} b^i_j \right\| \leq 2 \left\| f_i(t) b^i_i \right\| \sum_{i: r/2 < D(i,0) \leq r} \sum_{j: D(j,0) > r} h_{ij} b^i_j \leq \sqrt{\sum_{i: r/2 < D(i,0) \leq r} \left\| f_i(t) \right\|^2 n \frac{\max_{i: r/2 < D(i,0) \leq r} \sum_{j: D(j,0) > r} n}{D(i,j)\alpha}} \leq C_2 n^{3/2} t^{1/2} r^{(\alpha-d-\epsilon)/2},
\]
for \( 0 < C_2 < \infty \). Replacing Eqs. (148) and (149) into Eq. (145) and integrating over time, we arrive at Eq. (144).

5.4. Single particle state transfer

Our next goal is to prove the tightness of Theorem 9, up to subalgebraic corrections. This is achieved by the following theorem, which provides a rapid state-transfer protocol for a single particle.
FIG. 5. In this $d = 1$ illustration, first we iteratively redistribute the state evenly to doubled number of sites by $U_3 U_2 U_1$. Second, we translate the radius $O(r)$ balls across distance $r$. Third, we invert the first stage. The time and space scales are accurate in the sense that both the radius of infected region $B_i$ and duration of $U_i$ grows exponentially with $i$. The total duration is dominated by the translation stage, which uses a Hamiltonian of strength $O(\frac{1}{\alpha - d})$. Note the positions of balls $B_i$ are supposed to be around the same site, which shifts in this figure.

**Theorem 11.** For every $x \in \Lambda - \{0\}$ with $\mathcal{D}(x, 0) > 2$, there exists a constant $0 < K < \infty$ and a Hermitian matrix $h(t) : \mathbb{R} \to \mathbb{C}^{\Lambda \times \Lambda}$ obeying Eq. (14), such that $\langle x | U(\tau) | 0 \rangle = 1$ at

$$
\tau := K \times \begin{cases} 
\frac{\mathcal{D}(x, 0)}{\mathcal{D}(x, 0)^{\alpha - d}} & \alpha \geq d + 1 \\
\log \mathcal{D}(x, 0) & d < \alpha < d + 1 \\
1 & \alpha < d 
\end{cases}.
$$

(150)

Proof. First, suppose that $\alpha \geq d + 1$. Let $(y_0 := 0, y_1, \ldots, y_{\ell-1}, y_{\ell} := x)$ be a sequence of length $1 + \mathcal{D}(x, 0)$ such that the edge $(y_i, y_{i+1})$ is an edge of nearest neighbors in $\Lambda$; here $\ell := \mathcal{D}(x, 0)$. Then we choose the constant

$$
K = \frac{\pi}{2h},
$$

(151)

where $h$ is defined in Eq. (14), and choose

$$
H(t) := \begin{cases} 
\begin{cases} 
ih |y_j\rangle\langle y_{j-1}| - ih |y_{j-1}\rangle\langle y_j| & t \in [(j - 1) \frac{\pi}{2h}, j \frac{\pi}{2h}) \\
0 & \text{elsewhere}
\end{cases} & \alpha \geq d + 1 \\
\log \mathcal{D}(x, 0) & d < \alpha < d + 1 \\
1 & \alpha < d
\end{cases}.
$$

(152)

For $t \in [0, \frac{\pi}{2h}]$, a direct calculation leads to

$$
|\psi(t)\rangle = \cos(ht)|0\rangle + \sin(ht)|y_1\rangle;
$$

(153)

since at $t = \frac{\pi}{2h}$, the state is perfectly transferred to state $|y_1\rangle$, it immediately follows that

$$
|\psi(t)\rangle = \cos\left(\frac{\pi}{2h} \left(t - \frac{j\pi}{2h}\right)\right) |y_j\rangle + \sin\left(\frac{\pi}{2h} \left(t - \frac{j\pi}{2h}\right)\right) |y_{j+1}\rangle, \quad 0 \leq j < \ell, \quad t \in \left[\frac{\pi j}{2h}, \frac{\pi (j + 1)}{2h}\right].
$$

(154)

Hence perfect state transfer is achieved according to Eq. (150) with

$$
K = \frac{\pi}{2h}.
$$

(155)

Then, suppose that $\alpha \leq d + 1$. In this case, define $q \in \mathbb{Z}^+$ as

$$
q := \lfloor \log_2 \mathcal{D}(x, 0) \rfloor.
$$

(156)
Note that \( q \geq 1 \). Our strategy of state transfer is depicted in Figure 5. For \( s \in \mathbb{Z}^+ \), define

\[
B^q_s := \{ i \in \Lambda : \mathcal{D}(i, y) \leq 2^{s-1} \}.
\]

First, we expand the state \( |\psi(t)\rangle \) to be coherent uniform superpositions on all lattice sites within the balls \( B^0_1, B^0_2, \ldots, B^0_q \). Secondly, we translate the superposition on \( B^0_q \) to a superposition on \( B^q_2 \). Lastly, we reverse the first step, transferring the superposition on \( B^q_2 \) to \( B^{q-1}_q, \ldots, B^1_1 \) and finally on to \( |x\rangle \).

To calculate \( \tau \), we invoke the following Lemma:

**Lemma 12.** Let \( A \) and \( B \) be two disjoint subsets of \( \Lambda \), and \( 0 < C < \infty \) be chosen such that for all \( i \in A \) and \( j \in B \), \( |h_{ij}(t)| = C \) is consistent with Eq. (14). Then if

\[
|\psi(0)\rangle = \frac{1}{\sqrt{|A|}} \sum_{i \in A} |i\rangle,
\]

there exists a protocol \( H(t) \) such that for any \( \theta \in \mathbb{R} \),

\[
|\psi(T)\rangle \propto \frac{\cos \theta}{\sqrt{|A|}} \sum_{i \in A} |i\rangle + \frac{\sin \theta}{\sqrt{|B|}} \sum_{i \in B} |i\rangle,
\]

at time

\[
T \leq \frac{\pi}{2C \sqrt{|B||A|}}.
\]  

**Proof.** Choose the protocol

\[
H(t) := \text{sgn}(\tan \theta) \sum_{k \in A, j \in B} i C (|j\rangle \langle k| - |k\rangle \langle j|).
\]

Without loss of generality, we take \( \theta \in [0, \frac{\pi}{2}] \); the generalization to other \( \theta \) is straightforward. By permutation symmetry, the wave function takes the form Eq. (159) with \( \theta(t) \) a function of time. Pick any \( j \in B \). We can explicitly evaluate

\[
\frac{d \theta}{dt} = \frac{\sqrt{|B|}}{\cos \theta} \frac{d(j|\psi(t)\rangle)}{dt} = -i \frac{\sqrt{|B|}}{\cos \theta} (j|H|\psi(t)) = C \sqrt{|B||A|}.
\]  

Since the value of \( \theta \) at which \( |\psi(t)\rangle \) is given by Eq. (159) is in \([0, \frac{\pi}{2}]\), we conclude that Eq. (162) implies Eq. (160). \( \square \)

By construction, the time \( \tau \) of our perfect state transfer algorithm is given by

\[
\tau \leq 2 \sum_{n=1}^q T_{B,n} + T_{\text{long}}.
\]  

The first factor comes from the state transfer time \( T_{B,n} \) between \( B^0_{n-1} \) and \( B^0_n \) and \( B^q_{n+1} \) and \( B^q_n \), which are identical; the second factor represents the transfer time \( T_{\text{long}} \) from \( B^0_q \) to \( B^q_2 \). To evaluate these times, we note that there exists a constant \( 0 < a < \infty \) such that for all \( k \in \Lambda \),

\[
|B^k_n| \leq 2^{dn} a.
\]  

By definition, if \( i \in B^k_{n-1} \) and \( j \in B^k_n \), \( \mathcal{D}(i, j) < 2^{n+1} \). By Lemma 12 and Eq. (164),

\[
T_{B,n} \leq \frac{\pi}{2 \sqrt{|B^k_{n-1}||B^k_n|}} \left( \inf_{i \in A, j \in B} \frac{h}{\mathcal{D}(i, j)^a} \right)^{-1} \leq \frac{\pi}{2^{1+d(n+\frac{1}{2})-(n+1)\alpha} ah} := 2^{n(a-d)} A,
\]

for a constant \( 0 < A < \infty \). Similarly, from Eq. (156), if \( i \in B^q_0 \) and \( j \in B^q_r \), \( \mathcal{D}(i, j) < 2^{2+r} \); repeating the above argument leads to

\[
T_{\text{long}} \leq 2^{(q+2)(\alpha-d)} A'.
\]
for a constant $0 < A’ < \infty$. Defining $A_* = \frac{1}{2}\max(A, A’)$, we conclude that when $\alpha \neq d$

$$\tau \leq A_* \sum_{n=1}^{g+2} 2^n(\alpha-d) = A_* \frac{2^{(g+3)(\alpha-d)} - 1}{2^{\alpha-d} - 1}. \quad (167)$$

If $\alpha < d$, we find

$$\tau < \frac{A_*}{1 - 2^{\alpha-d}}, \quad (168)$$

is independent of $q$ and hence $D(x, 0)$. If $\alpha > d$, we find

$$\tau \leq A_* \frac{2^{3(\alpha-d)}}{2^{\alpha-d} - 1} 2^q(\alpha-d) \leq A_* \frac{2^{3(\alpha-d)}}{2^{\alpha-d} - 1} D(x, 0)^{\alpha-d}. \quad (169)$$

Finally, when $\alpha = d$,

$$\tau \leq A_* \sum_{n=1}^{g+2} 1 \leq A_* (2 + \log_2 D(x, 0)). \quad (170)$$

Each case above is consistent with Eq. (150).

There are two important consequences of Theorem 11. Firstly, even a single quantum mechanical degree of freedom can perform state transfer as asymptotically well as the best known protocol in an interacting many-body system [17] for $\alpha \geq d$. It is an interesting open question whether an asymptotically faster protocol exists for interacting quantum systems. Secondly, Theorem 11 proves that any possible improvement to Theorem 9 must be sub-algebraic. Both the linear light cone and the superlinear polynomial light cones we have proved for free quantum systems with long-range interactions are now known to be optimal. Theorem 11 is also applicable to spin systems, since the spin degrees of freedom may be treated as hardcore bosons. Similarly, the protocol applies to Hamiltonians with on-site and particle number conserving interactions such as the Bose-Hubbard model: the interactions have no effect since at all times during the protocol there is at most a single particle in the system.

5.5. Efficient early time classical boson sampling

The boson sampling problem was proposed by Aaronson and Arkhipov [21] as a potential candidate for the demonstration of quantum supremacy. While simulating the dynamics of bosons hopping on a lattice is generally a difficult task for classical computers, early-time evolutions where the bosons do not have enough time to hop too far from their initial positions can be simulated efficiently [28–30]. In particular, Ref. [28] considered a scenario where bosons were initially located at equal distances on a lattice and allowed to move in the lattice using only nearest-neighbor hoppings. Using the Lieb-Robinson bounds, the authors constructed an early-time classical sampler that efficiently captures the dynamics of the bosons up to time $t_*$ that scales polynomially with the system size and thereby demonstrated a dynamical phase transition in the computational complexity.

The early-time classical sampler was later generalized to more complicated systems with power-law hoppings [30]. However, the easiness timescale $t_*$ in this case only scales polynomially with the system size for $\alpha > 2d$ and scales logarithmically with the system size when $d + 1 < \alpha < 2d$. In this section, we show that the tight free-particle bound in this paper immediately imply that $t_*$ scales polynomially with the system size for all $\alpha > d$, i.e. an exponentially longer easiness timescale in the regime $\alpha \in (d, 2d]$ compared to the previous results [30].

For pedagogical reasons, we only describe here the high-level ideas behind the construction of the early-time boson sampler and argue for its efficiency using the technical results of Ref. [30]. We consider $N$ bosons hopping on a $d$-dimensional lattice under the Hamiltonian

$$H(t) = \sum_{i,j} J_{i,j}(t)b_i^\dagger b_j, \quad (171)$$

where $b_i$ is the bosonic annihilation operator on site $i$, $J_{i,j}(t) \leq 1/D(i, j)^\alpha$ are the hopping strengths, and the sums are over all sites $i, j$ on the lattice. We assume that the lattice has $M \propto N^\beta$ sites in total, where $\beta \geq 1$ is a constant.
The bosons are initially located on evenly spaced sites on the lattice so that the minimum distance between nearest occupied sites is \( 2L \propto (M/N)^{1/d} \propto N^{(\beta-1)/d} \) as shown in Figure 6. Denote these initial positions by \( j_1, \ldots, j_N \). We can write the initial state in terms of the creation operators:

\[
|\psi(0)\rangle = \prod_{k=1}^{N} b^\dagger_{j_k} |\text{vac}\rangle,
\]

where \( |\text{vac}\rangle \) is the vacuum state.

The aim of boson sampling is to sample the positions of the bosons at a later time \( t \). The idea of the early-time boson sampler in Refs. [28, 30] is that each boson primarily hops within its causal light cone, i.e. a bubble of radius \( r(t) \) centered on its initial position. For small enough time, \( r(t) < L \) and the bosons do not interfere with each other. The state of the system at this time can be approximated by a product state over the bubbles and therefore the positions of the bosons can be efficiently simulated by simulating the dynamics of each boson independently.

Let \( U(t) = T \exp(-i \int_0^t ds H(s)) \) be the evolution unitary generated by \( H \) at time \( t \). By inserting pairs of \( 1 = U^\dagger U \) in between the creation operators, the state of the system at time \( t \) can be written as

\[
|\psi(t)\rangle = \prod_{k=1}^{N} U(t)b^\dagger_{j_k} U^\dagger(t) |\text{vac}\rangle.
\]

Here, the evolution of the state can be simplified into independent evolutions of the creation operators \( b^\dagger_{j_k}(t) \equiv U(t)b^\dagger_{j_k} U^\dagger(t) \). Using our free-particle bound in Theorem 9, we can approximate \( b^\dagger_{j_k}(t) \) by its evolution within a light cone originated from \( j_k \):

\[
b^\dagger_{j_k}(t) \approx U_k(t)b^\dagger_{j_k} U^\dagger_k(t) \equiv \tilde{b}^\dagger_{j_k}(t),
\]

where \( U_k(t) = T \exp[-i \int_0^t ds H_k(s)] \) and \( H_k \) is the Hamiltonian constructed from \( H \) by taking only the hoppings between sites that are at most a distance \( L \) from \( j_k \). Using Corollary 10, the error of this approximation is \( O((Nt)^{3/2}/L^{(\alpha-d-\epsilon)/2}) \), where \( \epsilon \) is an arbitrarily small positive constant and we have assumed \( t \geq 1 \) without loss of generality. Repeating the approximations for all \( k = 1, \ldots, N \), we thereby show that the state \( |\psi(t)\rangle \) is approximately \( |\phi(t)\rangle = \prod_k \tilde{b}_{j_k}(t) |\text{vac}\rangle \).

Since the operators \( \tilde{b}_{j_k}(t) \) are supported on distinct regions, the bosons from different regions do not interfere with each other. Therefore the probability distribution for the positions of the bosons in \( |\phi(t)\rangle \) is simply the product of probability distributions of each boson hopping independently. Thus, at later time, the positions of the bosons in \( |\phi(t)\rangle \) can be efficiently sampled on a classical computer.
Note that the state $|\psi(t)\rangle$ only approximates $|\psi(t)\rangle$ up to some time $t_*$. To estimate $t_*$, we calculate the total error of the approximation. A simple calculation [30] shows that the total error of approximating the $N$ original bosons $\{b^\dagger(t)\}$ by the confined ones $\{\tilde{b}^\dagger(t)\}$ would be $O\left(N^{5/2}t^{3/2}/L^{(\alpha-d-\epsilon)/2}\right)$—$N$ times the error of approximating each $b^\dagger(t)$ by the corresponding $\tilde{b}^\dagger(t)$.

Requiring that the total error of the approximation is at most a small constant, we obtain

$$t_* \propto L^{\frac{\alpha-d-\epsilon}{3}} N^{-\frac{5}{3}} \propto N^{(1/3)(\alpha-d-\epsilon)-\frac{5}{3}},$$

(175)

where we have replaced $L \propto N^{(\beta-1)/d}$ from our assumption. Therefore, by choosing a small enough $\epsilon$, the easiness time $t_*$ increases polynomially with $N$ for all $\alpha > d(1 + \frac{5}{2\beta-1})$. In particular, the condition becomes $\alpha > d$ in the limit of large $\beta$. Therefore, our free-particle bound has improved the easiness time $t_*$ exponentially compared to Ref. [30] in the regime $\alpha \in (d, 2d]$.

6. GENERATING TOPOLOGICALLY ORDERED STATES

In this section, we study the minimum time it takes to create topologically ordered states from topologically trivial ones. Before we present our result, we shall define topologically ordered states and topologically trivial states following the definitions in Refs. [23, 31]. Suppose that the finite lattice $\Lambda$ has diameter $L$ and consists of $O(L^d)$ sites. Let $\{|\psi_1\rangle, \cdots, |\psi_k\rangle\}$ be a set of orthonormal quantum many-body states and define

$$\epsilon = \sup_\mathcal{O}\max_{1 \leq i,j \leq k} \{|\langle \psi_i | \mathcal{O} | \psi_i \rangle - \langle \psi_j | \mathcal{O} | \psi_j \rangle|, 2 \langle \psi_i | \mathcal{O} | \psi_j \rangle\},$$

(176)

where the supremum is taken over unit-norm operators $\mathcal{O}$ supported on a subset of the lattice with diameter $l \leq L/2$. Roughly speaking, $\epsilon$ quantifies the ability to distinguish between the states $\{|\psi_1\rangle, \cdots, |\psi_k\rangle\}$ using observables that are supported on only a fraction of the lattice. We say that the states are topological if there exist constants $c, \beta > 0$ such that $\epsilon \leq c L^{-\beta}$, and are trivial if $\epsilon$ is independent of $L$ [32]. We now use the Lieb-Robinson bound to bound the minimum time it takes to convert between topological and trivial states.

**Proposition 13.** Consider a time-dependent Hamiltonian $H$ with long-range interactions of exponent $\alpha$ on $\Lambda$. Let $U(t)$ be the evolution unitary generated by $H$ at time $t$, let $\{|\psi_1\rangle, \cdots, |\psi_k\rangle\}$ be a set of topologically ordered states, and let $\{|\phi_1\rangle, \cdots, |\phi_k\rangle\}$ be a set of topologically trivial states. If $\alpha > 2d + 1$ and there is a time $0 < \tau < \infty$ such that $|\psi_i\rangle = U(\tau)|\phi_i\rangle$, then there exists an $L$-independent constant $0 < K < \infty$ such that $\tau > K \tau^*$, where

$$\tau^* := \begin{cases} L & \text{when } \alpha > 3d + 1, \\ L^{\frac{2d}{\alpha+2d}} / \log^2 L & \text{when } 2d + 1 \leq \alpha \leq 3d + 1. \end{cases}$$

(177)

**Proof.** Consider an arbitrary operator $\mathcal{O}$ with a support diameter of $\ell \leq L/2$ and let $\mathcal{O}(t) = U(t)\mathcal{O}U^\dagger(t)$ be the evolved version of $\mathcal{O}$. We further introduce $\mathcal{O}(t, l') = \text{tr}_{B_{l'}} \mathcal{O}(t)$ as the version of $\mathcal{O}(t)$ truncated to a ball $B_{l'}$ of diameter $l' > l$ such that $l' - l$ is of order $L$. Using the triangle inequality, we have

$$|\langle \phi_i | \mathcal{O}(\tau, l') | \phi_i \rangle - \langle \phi_j | \mathcal{O}(\tau, l') | \phi_j \rangle| \leq 2 \|\mathcal{O}(\tau) - \mathcal{O}(\tau, l')\| + |\langle \phi_i | \mathcal{O}(\tau) | \phi_i \rangle - \langle \phi_j | \mathcal{O}(\tau) | \phi_j \rangle|.$$  

(178)

By our assumptions on the topological order of $|\psi_k\rangle$ and absence of topological order in $|\phi_k\rangle$, there exist constants $0 < \beta, a_{1,2} < \infty$ such that

$$a_1 - \frac{a_2}{L^\beta} \leq 2 \|\mathcal{O}(\tau) - \mathcal{O}(\tau, l')\|.$$  

(179)

On the other hand, using Proposition 2 for $\alpha > 2d + 1$ and $\tau < L/\bar{v}$, where $\bar{v}$ is a constant, there exists $0 < C_{1,2} < \infty$ such that

$$\|\mathcal{O}(t) - \mathcal{O}(\tau, l')\| \leq C_1 L^d \frac{\tau^{d+1} \log^2 l'}{l^{d-2d}} = C_2 \tau^{d+1} \log^2 l' L^{d-2d},$$

(180)

where the factor $L^d$ accounts for the support size of $\mathcal{O}$. For all $\alpha > 2d + 1$, Eq. (180) vanishes as $L$ increases, in contradiction with Eq. (179), unless $\tau = O(\tau^*)$. This completes the proof. \(\square\)
7. CLUSTERING OF CORRELATIONS

In addition to the dynamics of quantum systems, the Lieb-Robinson bounds also have implications for the eigenstates of a Hamiltonian. In Ref. [8], the authors show that if a time-independent power-law Hamiltonian with an exponent $\alpha$ has spectral gap $\Delta > 0$, the correlations between distant sites in the ground state of the system also decay with the distance as a power law with an exponent lower bounded by

$$\alpha' = \frac{\alpha}{1 + \tilde{v}\Delta^{-2}},$$

(181)

where $\tilde{v}$ is a constant that depends on $\alpha$.

The bound in Ref. [8] has a undesirable feature: for a given value of $\alpha$, varying the gap $\Delta$ also changes the minimum exponent $\alpha'$. Although this leads to an intuitive implication that larger energy gaps result in faster correlation decay, there is no known example where ground state correlations decay at a slower rate than a power law with an exponent $\alpha$. Indeed, we shall show that the cause of this problem is tied to the previous lack of an algebraic light cone in the quench dynamics. In particular, by using the Lieb-Robinson bounds with algebraic light cones [9, 14, 15, 33, 34], we show for all $\alpha > 2d$ that the ground state correlations must decay as a power law with the exponent lower bounded by the exponent of the Hamiltonian.

**Proposition 14.** Let $H$ be a power-law Hamiltonian with an exponent $\alpha$; let $A, B$ be local operators obeying $||A||, ||B|| \leq 1$, supported on $X, Y$ such that $|X| = |Y| = 1$ and $D(X, Y) = r > 0$. Assume that $H$ has a unique ground state $|\psi_0\rangle$ and spectral gap $\Delta$ to the first excited state. Define $C(r) := \langle \psi_0 | AB | \psi_0 \rangle - \langle \psi_0 | A | \psi_0 \rangle \langle \psi_0 | B | \psi_0 \rangle$ to be the connected correlator between $A, B$ in the ground state. Then

$$|C(r)| \leq \left[ \frac{2^{\gamma - 1} c \Gamma(\frac{\gamma}{2})}{\pi} \frac{\alpha^{\gamma/2}}{\Delta r} + 1 \right] \frac{\log^{\gamma/2} r}{r^\alpha},$$

(182)

where $c$ is a constant independent of $\alpha$, $\gamma = \alpha(\alpha - d + 1)/(\alpha - 2d)$, and $\Gamma(\cdot)$ is the Gamma function.

**Proof.** First we rewrite

$$C(r) = \sum_{k \geq 0} \langle \psi_0 | A | \psi_k \rangle \langle \psi_k | B | \psi_0 \rangle,$$

(183)

where the sum is over the excited states $|\psi_k\rangle$ of the Hamiltonian. Our strategy is to relate $C(r)$ to the commutator norm $||[A(t), B]||$, which we then bound using a Lieb-Robinson bound. To relate $C(r)$ to $||[A(t), B]||$, it is natural to first consider the value of $[A(t), B]$ in the ground state, whose magnitude is bounded by $||[A(t), B]||$:

$$\langle \psi_0 | [A(t), B] | \psi_0 \rangle = \langle \psi_0 | A(t) B | \psi_0 \rangle - \text{h.c.} = \sum_{k > 0} e^{i E_k t} \langle \psi_0 | A | \psi_k \rangle \langle \psi_k | B | \psi_0 \rangle - \text{h.c.},$$

(184)

where $E_k$ are the eigenvalues of the Hamiltonian and we have set ground state energy $E_0 = 0$ so that $E_k > 0$ for all $k > 0$. Note that the $k = 0$ terms cancel between the first term and its Hermitian conjugate.

By observation, we note that if we could replace the terms $e^{i E_k t}$ in Eq. (184) by a unit step function $\Theta(E_k)$ that satisfies $\Theta(E_k) = 1$ and $\Theta(-E_k) = 0$, we immediately obtain the expression of $C(r)$ in Eq. (183). In fact, this replacement is easy to achieve using the identity

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{e^{iE_k t}}{t - i\epsilon} = \Theta(E_k).$$

(185)

Therefore, we have

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{\langle \psi_0 | [A(t), B] | \psi_0 \rangle}{t - i\epsilon} = C(r),$$

(186)

and we obtain the relation

$$|C(r)| = \left| \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{\langle \psi_0 | [A(t), B] | \psi_0 \rangle}{t - i\epsilon} \right| \leq \frac{1}{\pi} \int_{0}^{\infty} dt ||[A(t), B]|| \frac{1}{t}.$$
Unfortunately, this relation is not useful; the right-hand side of Eq. (187) diverges even when the commutator $\|[A(t), B]\|$ does not increase with time. The failure of such a simple treatment is not surprising as we have not used the crucial assumption on the existence of a finite energy gap ($E_k \geq \Delta$).

Intuitively, to make the integral in Eq. (187) converge, we can multiply the integrand by a function that decays quickly with $t$. A natural choice is a Gaussian function $e^{-(vt/2)^2}$, where $v > 0$ is an adjustable parameter; it decays with time quickly enough to make the integral converge and its Fourier transformation is rather easy to handle. By multiplying this function to the integrand in Eq. (185), we arrive at a convolution of the step function with the Gaussian function:

$$
\lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \frac{e^{iE_k t} e^{-(vt/2)^2}}{t - i\epsilon} = \frac{1}{\sqrt{\pi} v} \int_{-\infty}^{\infty} \Theta(E_k - E) e^{-E^2/2v^2} dE =: f(E_k). \quad (188)
$$

It is easy to verify that $f(E_k) = 1 - g(E_k)$ and $f(-E_k) = 0 + g(E_k)$ for some positive function $g(E_k) \leq \frac{1}{2} e^{-(E_k/v)^2}$. Thus, $f(E_k)$ closely resembles the step function $\Theta(E_k)$, albeit with a smoother transition from 0 to 1.

Inserting this convolution into Eq. (186), we have:

$$
\lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \langle \psi_0 | [A(t), B] | \psi_0 \rangle e^{-(vt/2)^2} = C(r) - \sum_{k>0} g(E_k) |\langle \psi_0 | A | \psi_n \rangle \langle \psi_n | B | \psi_0 \rangle + {\mathrm{h.c.}}| \quad (189)
$$

Using a Cauchy-Schwarz inequality, we can then bound the absolute value of the sum over $k$ in the right-hand side by

$$
\sum_{k>0} 2g(E_k) |\langle \psi_0 | A | \psi_n \rangle \langle \psi_n | B | \psi_0 \rangle| \leq e^{-(-\Delta/v)^2}, \quad (190)
$$

where we have used that $E_k \geq \Delta$. Thus we arrive at our desired relation:

$$
|C(r)| \leq \frac{1}{\pi} \int_0^{\infty} dt \frac{e^{-(vt/2)^2}}{t} \|[A(t), B]\| e^{-(-\Delta/v)^2}. \quad (191)
$$

Finally, we bound the commutator norm using the Lieb-Robinson bound in Ref. [9],

$$
\|[A(t), B]\| \leq \frac{t^\gamma}{r^\alpha}, \quad (192)
$$

where $c$ is a constant and $\gamma = \alpha(\alpha - d + 1)/(\alpha - 2d)$. We obtain:

$$
|C(r)| \leq \frac{2^{\gamma - 1} \Gamma(\frac{\gamma}{2})}{\pi} \frac{1}{v^{\gamma} r^\alpha} + e^{-(-\Delta/v)^2}, \quad (193)
$$

where $\Gamma(\cdot)$ is the Gamma function. By choosing $v = \Delta/\sqrt{\alpha \log r}$, we get

$$
|C(r)| \leq \left[ \frac{2^{\gamma - 1} \Gamma(\frac{\gamma}{2})}{\pi} \frac{\alpha^{\gamma/2}}{\Delta^\gamma} + 1 \right] \frac{\log^{\gamma/2} r}{r^\alpha}. \quad (194)
$$

Therefore, the correlators in the ground state of a power-law Hamiltonian with $\alpha > 2d$ also decay with the distance as a power law (up to a logarithmic correction) with the same exponent $\alpha$ as that of the Hamiltonian. In particular, this exponent is independent of the energy gap $\Delta$, in contrast to the previous result in Ref. [8].

Note that in Eq. (193), we have used an algebraic light cone bound from [9] instead of the tighter bounds in recent works [14, 15, 33, 34], because the bounds in Refs. [14, 33, 34] decay with the distance slower than $1/r^\alpha$ while the bound in Ref. [15] does not hold for $2d < \alpha \leq 2d + 1$.

8. CONCLUSION

We have demonstrated a hierarchy of linear light cones—a sequence of metrics and protocols under which the emergent locality that arises in local quantum many-body systems breaks down at different exponents $\alpha$ of long-range
interactions. The most general such light cone—the Lieb-Robinson light cone that bounds commutator norms—can become superlinear for any $\alpha < 2d + 1$. We conjecture that the Frobenius light cone that controls many-body chaos and state transfer can only be superlinear when $\alpha < 1 + \frac{3}{2}d$, and proved this result in $d = 1$ using the operator quantum walk formalism. Finally, in non-interacting systems, we proved both linear ($\alpha > d + 1$) and superlinear ($d < \alpha \leq d + 1$) light cones along with the optimality of these bounds. As such, we close a number of long-standing questions in the literature on the limits and capabilities of quantum dynamics with long-range interactions.

Besides state transfer and many-body chaos, we have also demonstrated a wide range of applications of these (nearly) tight light cones. We proved that the growth of connected correlations obey the same light cone as that of the Lieb-Robinson bound. In the context of digital quantum simulation, we used the Lieb-Robinson bound to construct an approximation for the time-evolved version of a local observable, and thereby reduced cost of simulating the observable on quantum computers for all $\alpha > 2d + 1$. Similarly, using the free light cone, we constructed an efficient early-time classical boson sampler for all $\alpha > d$, exponentially improving the previous best estimate in some regime of $\alpha$. Additionally, we bounded the time it takes to generate topologically ordered states using power-law interactions. Finally, we tightened the minimum correlation-decay rate in the ground state of a gapped power-law Hamiltonian.

The hierarchy of linear light cones revealed in this paper has important implications both on the capabilities of quantum technologies exploiting long-range interactions, as well as on the nature of quantum information dynamics and thermalization in these systems. A complete understanding of quantum chaos and state transfer, at the very least, requires the construction of a new mathematical framework beyond the Lieb-Robinson bounds, perhaps along the lines of our operator quantum walk. It remains an important future challenge to obtain the Frobenius light cone in two or more dimensions, as well as to rigorously study the light cone that controls the decoherence of a quantum system subject to long-range random noise, which was conjectured to be linear for $\alpha > d + \frac{1}{2}$ [35].

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