Entanglement Bounds on the Performance of Quantum Computing Architectures

Zachary Eldredge,1, 2 Leo Zhou,3 Aniruddha Bapat,1, 2 James R. Garrison,1, 2
Abhinav Deshpande,1, 2 Frederic T. Chong,4 and Alexey V. Gorshkov1, 2

1Joint Center for Quantum Information and Computer Science, NIST/University of Maryland, College Park, Maryland 20742, USA
2Joint Quantum Institute, NIST/University of Maryland, College Park, Maryland 20742, USA
3Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA
4Department of Computer Science, University of Chicago, Chicago, Illinois 60637, USA

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There are many possible architectures for future quantum computers that designers will need to choose between. However, the process of evaluating a particular connectivity graph’s performance as a quantum architecture can be difficult. In this paper, we establish a connection between a quantity known as the isoperimetric number and a lower bound on the time required to create highly entangled states. The metric we propose counts resources based on the use of two-qubit unitary operations, while allowing for arbitrarily fast measurements and classical feedback. We describe how these results can be applied to the evaluation of the hierarchical architecture proposed in Phys. Rev. A 98, 062328 (2018). We also show that the time-complexity bound we place on the creation of highly-entangled states can be saturated up to a multiplicative factor logarithmic in the number of qubits.

As the development of quantum computers progresses from the construction of qubits to the construction of intermediate-scale devices, quantum information scientists have increasingly begun to explore various architectures for scalable quantum computing [1–4]. Researchers have quantified the cost imposed by moving from one architecture to another [5, 6] and optimized the placement of qubits on a fixed architecture [7–9]. Experimentalists have also begun to test different architectures in laboratory settings [10, 11].

In this work, we are interested in developing tools to evaluate the relative performance of different architectures. Here, “architecture” refers to the connectivity graph that defines the allowable elementary operations between qubits. We propose a natural metric based on entanglement measures. When several physical models are represented by a graph \( G = (V, E) \), with a set of vertices \( V \) corresponding to qubits, and a set of weighted edges \( E \) corresponding to two-qubit interactions, a useful metric is given by what we dub the “rainbow time,”

\[
\tau_{RB}(G) = \max_{F \subseteq V : |F| \leq \frac{1}{4}|V|} \left[ \frac{|F|}{|\partial F|} \right],
\]

where \(|\partial F|\) denotes size of the boundary of \( F \), i.e. the total weight of edges connecting \( F \) and \( V - F \).

The rainbow time is the minimum time required to create a large entangled state on the graph (states of \( N \) qubits with bipartite entanglement \( O(N) \)). It is also the inverse of a well-studied graph quantity known as the isoperimetric number [12]. In contrast to Ref. [13], where architectures are evaluated assuming that only unitary operations are permitted, using the creation of a GHZ state as a benchmark, we do not make this assumption here. Including operations like measurement and feedback allows the GHZ state to be created in unit time regardless of graph structure, making the GHZ state no longer a useful benchmark [14].

The rainbow time metric is a lower bound on the time required to create highly entangled states (states of \( N \) qubits with bipartite entanglement \( O(N) \)). As a complementary result, we show that this bound is nearly tight – a procedure which distributes Bell pairs using maximum-flow algorithms nearly saturates this bound, up to \( O(\log N) \) overhead.

Physical setup.— In this paper, we evaluate the performance of quantum architectures with a connectivity graph given by \( G \). We assume every graph introduced is weighted, connected, and has at least two vertices. Each vertex in the graph represents a single data qubit, and an edge exists between two vertices if elementary two-qubit gates can be performed between them. We interpret the edge weight \( w_{ij} \) between vertices \( i \) and \( j \) as representing bandwidth, so that higher-weighted edges are capable of performing more elementary two-qubit operations in a single unit of time.

In the main text, we will interpret the graph model (Fig. 1) as representing the rate of distribution of entangled pairs as in Ref. [15]. Each graph vertex contains one data qubit, which sits within a small module that includes ancillas serving as auxiliary communication qubits. In each unit of time \( T \), Bell pairs are generated on the edges of the graph, which can then be used to perform two-qubit gates [16, 17]. The process of moving from this model to an abstracted connectivity graph is illustrated in Fig. 1.

We assume that measurements, classical communication, and intra-module unitaries are arbitrarily fast, such that the bottleneck is given by quantum operations between modules. For example, the qubits could be part of a trapped-ion system which uses photonic interconnects.
to generate entanglement between models, as described in Refs. [18, 19]. In these systems gates between modules take much longer than operations that are limited to a single module. In this framework, vertex degrees and total graph edge weights represent required ancilla overheads, justifying their use as cost functions in Ref. [13]. In the Supplemental Material [20], we extend our results to cases in which edge weights represent the number of unitaries that can be executed in one time step or the local coupling strengths of a Hamiltonian system [20].

Entanglement capacity.—Given a graph \( G \), we wish to bound the total possible increase in a given entanglement measure after \( n \) rounds of entanglement distribution over its links. Suppose we fix a bipartition of the graph into two subgraphs, \( F \) and \( \bar{F} \). We consider a general entanglement measure, \( S \), which quantifies the bipartite entanglement between \( F \) and \( \bar{F} \). We assume that \( S \) is zero for product states \( \rho_F \otimes \rho_{\bar{F}} \), additive between nonentangled regions, \( S(\rho_F \otimes \tau_{\bar{F}}) = S(\rho_F) + S(\tau_{\bar{F}}) \), and that \( S \) is non-increasing under local operations and classical communication. Entanglement measures that obey these conditions include the entanglement cost, the distillable entanglement, and the entanglement of formation \([21, 22]\). All of these measures are identical to the von Neumann entropy for pure states.

By the result of Ref. [21], the entanglement after \( n \) rounds is bounded by \( n \) times the maximum single-round entanglement. We will therefore bound the entanglement generated in one round, going from \( \rho \) to \( \rho' \). To produce \( \rho' \), we begin with \( \rho \) and then generate entanglement on the graph edges. This means that \( w_{ij} \) ancilla Bell pairs are generated for each edge \((i, j)\) crossing the boundary \( \partial F \). The total number of Bell pairs is therefore \( |\partial F| \), the sum over all the weights,

\[
|\partial F| = \sum_{i \in F, j \in \bar{F}} w_{ij},
\]

(2)

Ignoring ancillas purely local to \( F \) or \( \bar{F} \), the resulting state is \( \rho \otimes \rho_{\bar{F}}^{\partial F} \). The final state \( \rho' \) is then generated by local operations, assisted by classical communication, on this state. We denote the state that results from an arbitrary round of local operations and classical communications on \( \rho \) as \( \text{LOCC}(\rho) \). Therefore, our axioms for \( S \) allow us to write:

\[
S(\rho') = S\left(\text{LOCC}\left(\rho \otimes \rho_{\bar{F}}^{\partial F}\right)\right),
\]

(3)

\[
\leq S(\rho \otimes \rho_{\bar{F}}^{\partial F})
\]

(4)

\[
= S(\rho) + |\partial F| S(\rho_{\bar{F}}),
\]

(5)

\[
\implies S(\rho') - S(\rho) \leq |\partial F| S(\rho_{\bar{F}}).
\]

(6)

We refer to this bound on the change in entanglement, \( \Delta S \leq |\partial F| \) in units where \( S(\rho_{\bar{F}}) = 1 \), as the entanglement capacity of the graph \( G \) across the \((F, \bar{F})\) bipartition.

Rainbow states.—We now define a highly entangled state that serves as a benchmark for the performance of a quantum computing architecture.

Entanglement makes a useful benchmark for any quantum computer because it can be shown that computations that do not produce entanglement can be simulated efficiently classically [23, 24]. Further motivation for producing highly entangled states can be found in quantum simulation, where a quantum simulator of general applicability ought to be capable of representing and simulating highly entangled states [25].

To select a particular entangled state, we first consider the “rainbow state” which is defined in 1D contexts for even \( N \) as one in which qubits \( i \) and \( N - i \) are maximally entangled [26, 27]. The state itself is maximally entangled across a bipartition between the first \( N/2 \) qubits and the rest.

We wish to extend this construction to arbitrary graphs. Suppose we consider a set of qubits \( V \) and any subset \( F \subset V \), with the requirement that \( |F| \leq \frac{1}{2} |V| \). Denote by \( F_i \) the \( i \)th vertex of \( F \) using an arbitrary ordering and similarly use \( \bar{F}_i \) to index vertices in the complement \( \bar{F} \). We can then define a “rainbow” state as one in which qubit \( F_i \) and qubit \( \bar{F}_i \) form a Bell pair and any additional qubits in \( \bar{F} \) are left in the state \( |0\rangle \). This state is illustrated for a particular arbitrary \( F \) in Fig. 2. Note that this construction is only well-defined if \( |F| \leq \frac{1}{2} |V| \), as otherwise there will not be enough data qubits in \( \bar{F} \) to form Bell pairs with all the data qubits in \( F \).

Rainbow times and isoperimetric number.—Using the model for quantum architectures in which the edge weights of a graph \( G \) denote the number of gates that can be performed, we can calculate the lower bound on the time required to create the rainbow state, according to the entanglement capacity. For any set \( F \) we define this time, \( t(F) \),

\[
t(F) = \frac{|F|}{\sum_{i \in F, j \in \bar{F}} w_{ij}},
\]

(7)
As we have shown, the boundary constrains the amount of entanglement that can be distributed to the subsystem $F$ from its complement $\bar{F}$ in a single time step.

While there is some freedom in choosing a highly entangled physical state associated with the set $F$ that would be hard to create, here we argue why the above metric $t(F)$ suffices for most considerations. Although there are many different states with $O(N)$ entanglement which could be used to evaluate graphs, the rainbow state is easy to illustrate and create. A rainbow state, as a set of Bell pairs, offers insight into the time required to create a general bipartite entangled state, since states can be converted either to or from Bell pairs through entanglement concentration or dilution [28]. Furthermore, the rainbow state arises as the ground state of some interesting models in condensed-matter physics, as shown in Ref. [29]. The rainbow state was also used as an example of a difficult-to-create state in Ref. [14]. To associate a physical rainbow state with the set $F$, we must specify how vertices in $F$ are paired with those in $\bar{F}$. However, that ordering does not affect the minimum time required to create the state according to the entanglement capacity, $t(F)$. While different rainbow states that share a common $F$ may differ substantially in how quickly they can be created, $t(F)$ serves as the common lower bound on creation time for all of them, and we will focus on that metric here.

We will now use $t(F)$ to evaluate the quantum architecture $G$, the larger graph that $F$ is a subgraph on. To do this, we find the maximum $t(F)$ given $G$. Note that this is not the same as maximizing entanglement entropy, which would simply yield half the graph without any consideration of the graph structure. Instead we ask: of all the maximally-entangled states we can build by bipartitioning $V$ into $F$ and $\bar{F}$, which of them is slowest to build according to the entanglement capacity? We call the associated $t(F)$ the rainbow time of the graph $G$ and denote it $\tau_{RB}(G)$, as defined in Eq. (1).

The rainbow time has a simple and attractive interpretation, can be directly connected to quantum computing tasks, and is applicable to various physical models of computation [20]. In addition, it can be directly connected to a quantity known as the isoperimetric number, $h(G)$ [12], which is well-studied in graph theory and computer science [30–32]. As we have defined it, the rainbow time is simply $\tau_{RB}(G) = 1/h(G)$ [33]. Thus, aiming to minimize the rainbow time (so that large entangled states can be easily created) in a quantum architecture is equivalent to maximizing the isoperimetric number. An “isoperimetric set” is a set $F$ that achieves $t(F) = \tau_{RB}(G)$. Often, isoperimetric numbers appear in the context of expander graphs, which are constructed to possess large isoperimetric numbers [34] and can be used to prove results in complexity theory [35–37]. Intuitively, a small isoperimetric number (large $\tau_{RB}$) means that a graph has bottlenecks, and a sizable subset can easily be disconnected by removing relatively few edges. This also implies that an architecture with large $\tau_{RB}$ is also more prone to becoming disconnected due to the failure of a small number of edges. In some literature, the isoperimetric number is referred to as the Cheeger constant.

In the Supplemental Material [20], we evaluate the rainbow time of a variety of graphs, such as complete, star, and grid graphs. Although the computation of exact rainbow time is NP-hard in general [12], we are able to calculate a rainbow time for these graphs as well as the hierarchical products and hierarchies presented in Ref. [13]. In particular, we compare hierarchies to $d$-dimensional grids and show that, for some parameters, hierarchies have lower rainbow time and lower total edge weight than grids, making hierarchies promising architecures for quantum computing. The rainbow time can also be bounded by using the eigenvalues of the graph Laplacian as well as other known bounds shown in Ref. [12].

Creating rainbow states.— So far we have shown that rainbow time $\tau_{RB}$ serves as a lower bound for generating maximum entanglement across any bipartition of the system. We now examine whether this bound can be saturated, in the sense that one can construct an explicit protocol to prepare a rainbow state across any bipartition in time $O(\tau_{RB})$. We will show that for a general graph, any rainbow state can be prepared in a time no more than $|F|\ln|F|$ for a bipartition where $F$ is the smaller subgraph. This suggests that in addition to providing a bound, the rainbow time can be a useful witness to the speed at which entanglement can actually be generated in a quantum architecture. The bound provided by $\tau_{RB}$ is tight up to a factor logarithmic in the size of the state being created.

We begin the proof by mapping the problem of rainbow state generation to the MaxFlow problem in computer science [38]. We begin by specifying some vertex.
FIG. 3. An illustration of the fictitious nodes added to the isoperimetric set, $F$ and a set of equal size $K$ (encircled by purple dashed line), to create a flow network. The new fictitious nodes, $s$ and $t$, appear as green triangles connected to every node in $F$ and $K$ respectively; the original nodes and edges are pink (in $F$) and blue (in $\tilde{F}$) circles. The edges have weight one. The flow, shown by arrows, transfers $|F|/\tau_{RB} = 2$ units of entanglement across the bipartition. Grey, dotted edges are not used by the flow.

In order to demonstrate that a flow approach yields an efficient creation of a rainbow state, we will make use of the fact that the maximum flow is of the same value as the minimum cut, known as the MaxFlow-MinCut theorem [38]. Here, a “cut” means a bipartition of the graph separating $s$ and $t$, and the value of the cut is the total weight of all edges that cross the bipartition. By finding a lower-bound on the value of all possible cuts in a graph, we show that a flow larger than or equal to this bound must exist.

Suppose that we now consider any cut of the graph into arbitrary pair of sets $\{s\} \cup S$ and $\{t\} \cup T$. The boundary of this cut will consist of edges from $s \rightarrow T$, $S \rightarrow t$, and $S \rightarrow T$. Its boundary can be written as

$$|\text{Cut}(S,T)| = |T \cap F| + |S \cap K| + |\partial S|, \quad (8)$$

since $s$ and $t$ are connected only to nodes in $F$ and $K$, respectively, and the edges in $S \rightarrow T$ are just the boundary of $S$ in the original graph. To evaluate $|\partial S| = |\partial T|$, we will assume that $|S| \leq \frac{1}{2} |V|$, meaning we can apply the isoperimetric condition $|S| \leq |\partial S| \tau_{RB}$. (If this is not the case, a near-identical argument can be made applying this condition to $T$.) To account for cases where $\tau_{RB} < 1$, we will write this as $|\partial S| \geq m |S|$ where $m = \min \left(1, 1/\tau_{RB}\right)$. We then note that:

$$|\partial S| \geq m |S| \geq m \left(|S \cap F| + |S \cap K|\right), \quad (9)$$

$$\geq m \left(|F| - |T \cap F| + |S \cap K|\right). \quad (10)$$

By inserting this lower bound for $|\partial S|$ into Eq. (8), we obtain

$$|\text{Cut}(S,T)| \geq (1 - m) |T \cap F| + (1 + m) |S \cap K| + m |F|. \quad (11)$$

Since we know $m \leq 1$, we obtain the final bound on the cut magnitude,

$$|\text{Cut}(S,T)| \geq m |F|. \quad (12)$$

If $m = 1$ (i.e., $\tau_{RB} \leq 1$), it follows that the value of the smallest cut is greater than $|F|$, meaning that a flow exists of magnitude at least $|F|$, which creates the rainbow state in a single round. If $m < 1$ (i.e., $\tau_{RB} > 1$), we find that a flow exists of magnitude $|F|/\tau_{RB}$ [41]. Once $|F|/\tau_{RB}$ nodes are entangled, they can be disconnected from $s$ and $t$ and the process repeated on a new set of nodes $F_1 \subset F$. Therefore after $n$ rounds of computation, the remaining set of nodes waiting for entanglement $F_n$ is produced by removing $1/\tau_{RB}$ of the nodes in set $F_{n-1}$, with $F_0 = F$, allowing us to compute the maximum size of $F_n$ inductively:

$$|F_n| \leq \left(1 - \frac{1}{\tau_{RB}}\right) |F_{n-1}| \quad (13)$$

$$\leq \left(1 - \frac{1}{\tau_{RB}}\right)^n |F| < e^{-n/\tau_{RB}} |F|. \quad (14)$$
Once $|F_n| < 1$, the process is complete, as there are no fractional nodes. It follows that $\lceil \tau_{RB} \ln |F| \rceil$ rounds suffice to complete the entangling process.

**Outlook.** — In this work, we have presented a new metric for evaluating proposed architectures for quantum computers. In the future, we would like to understand the limitations this metric has in some computational models. For some graphs, for instance, the star graph $S_N$, low rainbow time is possible only because of very high central node degree. Placing a limitation on the number of operations an individual node can perform in a time step, regardless of how many other nodes it is connected to, is an interesting problem that would require modifying the framework we have presented here.

While we presented a proof that any set $F$ can have a rainbow state prepared in $\lceil \tau_{RB} \ln |F| \rceil$ time, simulations consistently suggest that flow-based algorithms can create rainbow states in $\lceil \tau_{RB} \rceil$ time. It is possible that the logarithmic factor can be removed and that the rainbow time lower bound is tight and saturable. In addition, although our argument suggests that a rainbow state can be created in $\lceil \tau_{RB} \ln |F| \rceil$ time, a permutation of the node indices specifying another rainbow state on the same set of qubits may be more difficult to create. One way of understanding this problem might be to analyze graphs using tools from classical network theory such as routing time [42, 43].

Finally, another line of inquiry to pursue is how the entanglement capacity, which was used here in terms of the rainbow time, can be applied to the analysis of quantum algorithms. For instance, by showing that a certain amount of entanglement is required to perform other tasks in quantum computing, rainbow time could then serve as a lower bound on the time required to perform this task on a particular architecture. References [44, 45] explore the question of how entanglement grows during Shor’s algorithm and in adiabatic quantum computing. These complement other results which show that low-entanglement systems can be simulated efficiently on a classical computer [23, 46]. Rainbow time could also be a fruitful way to test algorithms for compilation and gate decomposition of quantum circuits, by comparing their realized circuit depth to this theoretical minimum required time.

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**References**

[20] Please see Supplemental Material online.
Note that, as we have defined it so far, $\tau_{RB}(G)$ can take on any nonnegative real value. In reality, the creation of a quantum state will always take an integer number of steps greater than or equal to one in our model. Therefore, $\lceil \tau_{RB} \rceil$ can be used as a measure of the “number of rounds required” in cases where this is important.
S1. ENTANGLEMENT CAPACITIES ON VARIOUS PHYSICAL MODELS

In this section we will derive the entanglement capacity for several different physical models that can correspond to a graph. Consider a graph, $G$, and select a subset of the graph, $F$. We then want to show that the maximum amount of entanglement that can be created between $F$ and $\bar{F}$ in unit time is proportional to the size of the boundary, $|\partial F|$. We will allow arbitrary constant factors, and discuss how this bound arises in two different physical situations. As in the main text, we consider entanglement measures $S$ on two regions so long as $S$ obeys the following rules:

- Additively distributive over the tensor product, so $S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$ if $\rho$ and $\sigma$ are supported on both sides of the bipartition.
- Zero for states which are a product of states on each region, $S(\rho_F \otimes \rho_{\bar{F}}) = 0$.
- Non-increasing after any operation which is local to each region, even if we permit classical communication.

In the main text, we showed how to apply these axioms to the analysis of a case in which computation was performed by the production and consumption of Bell pairs. Here we also look at a gate model of computation and a case in which the graph describes the limits on a time-dependent interaction Hamiltonian.

**Unitaries.** In this model, each graph edge of weight $w_{ij}$ represents the capability to perform $w_{ij}$ unitaries between qubits $i$ and $j$ in a time step. These unitaries are freely chosen by the experimenter. For two qubits, the ability to apply multiple unitaries is no different from the ability to apply an arbitrary unitary. However, we are considering cases where the qubits are part of a larger system, meaning we may wish to perform unitaries in sequence on different pairs to perform a more complicated computation.

We note that every two-qubit unitary can be performed using two Bell pairs as a shared resource and applying local operations. This can be easily seen in the following process:

1. Alice and Bob start with a data qubit each and two Bell pairs shared between them. They wish to implement an arbitrary two-qubit unitary using only local operations and classical control.
2. Alice uses one Bell pair and classical communication to teleport her qubit to Bob.
3. Bob uses his local operations to perform the desired two-qubit gate.
4. Bob teleports Alice’s qubit back to her.

Therefore, the state $\rho'$ can be obtained from the state $\rho$ by using LOCC and consuming up to $2|\partial F|$ Bell pairs in the process. Since LOCC cannot increase $S$, it follows that:

$$S(\rho') \leq S\left(\rho \otimes \rho_{\text{Bell}}^{\otimes 2|\partial F|}\right)$$

$$\implies \Delta S \leq 2|\partial F| S(\rho_{\text{Bell}}).$$

(S1)  (S2)

This suggests that the ability to perform arbitrary unitaries is up to twice as powerful as the ability to distribute arbitrary Bell pairs, which makes sense, as an arbitrary two-qubit gate cannot necessarily be performed with one Bell pair (for instance, SWAP requires two) [S1]. Two Bell pairs however suffice to implement any arbitrary two-qubit unitary. In any case, this still yields an entanglement capacity $\Delta S = O(|\partial F|)$ bound as desired.
Hamiltonians. We will now consider a case in which the graph describes a Hamiltonian, possibly time-dependent. The graph will restrict the strength of these Hamiltonians. If we assume that $G = (V, E)$, then the Hamiltonian can be written as a sum over the two-qubit operations:

$$ H(t) = \sum_{(i,j) \in E} h_{ij}(t). $$

(S3)

We then impose the condition:

$$ \forall t : ||h_{ij}(t)|| \leq w_{ij}, $$

(S4)

where $w_{ij}$ is the i-j edge weight.

We can then apply the “small incremental entanglement” (SIE) theorem [S2]. In particular, we apply the special case used in Ref. [S3] to bound the total amount of entanglement generated by this Hamiltonian. If $H$ is a sum of pairwise Hamiltonians $h_{ij}$ acting on qubits, then the time-rate of entanglement generation on a set $F$ of sites is:

$$ \left| \frac{dS_F}{dt} \right| \leq 36 \log(2) \sum_{i \in F, j \notin F} ||h_{ij}||. $$

(S5)

Here, $S_F$ is the von Neumann entropy of the reduced density matrix on the region $F$. This can be derived from Eq. (3) of Ref. [S3] and specifying two-body terms and qubit sites, but the result could be extended to qudits or to general k-body interactions with a different constant in front. The sum over Hamiltonian norms, in the graph context, corresponds to a sum over graph edges. Since every Hamiltonian strength is limited by the corresponding edge weight, $\sum ||h_{ij}|| \leq \sum w_{ij} = |\partial F|$. Therefore, we can specifically say that for this case, $\Delta S_F = O(|\partial F|)$. Many other entanglement measures, such as entanglement of formation or entanglement cost, can be related to the von Neumann entropy [S4]. In particular, many entanglement measures on mixed states can be defined as a weighted sum over pure state components; since none of the pure states can increase dramatically in entanglement under this process, the entanglement measure on the mixed state is similarly limited.

S2. APPLICATION TO HIERARCHICAL PRODUCT AND HIERARCHIES

In this section, we calculate the rainbow times for the hierarchical products and hierarchies of Ref. [S5]. A hierarchical product is a graph product denoted $G \Pi H$ in which $|G|$ copies of $H$ are connected at their root (first) vertices by the graph $G$. By iterating this process, we can create a hierarchy, in which higher-level graphs connect lower-level identical sub-hierarchies. We also extend this concept to that of a weighted hierarchy, in which the edges on level $i$ have weight $\alpha_i$. We write a $k$-level hierarchy with a vector of weights $\vec{\alpha}$ as $G^{\Pi \alpha_k}$, where $G$ is the base graph. Finally, if $\alpha_i = \alpha^{i-1}$, so that edge weight scales geometrically with the level of the hierarchy, we simply write $G^{\Pi \alpha_k}$. Some examples are shown in Fig. S1.

![Fig. S1. Examples of a hierarchical product (left) and a weighted hierarchy (right).](image)

In order to calculate the rainbow time for a hierarchical product, we make use of the result from Ref. [S6] that there must exist an isoperimetric set [a vertex set $F$ such that $\tau(F) = \tau_{RB}(F)$] that is connected and whose complement $\bar{F}$ is connected. Therefore, we will look at all possible subgraphs of $H_1 \Pi H_2$ where both $F$ and $\bar{F}$ are connected. From these, we will search for the one with the largest $\tau(F)$. Since some isoperimetric set is guaranteed to exist in this set of subgraphs, this maximization over $\tau(F)$ in this set will also give us $\tau_{RB}(H_1 \Pi H_2)$. We will begin by specifying
three cases, illustrated in Fig. S2. These cases cover all possible subsets with the right connectedness properties and therefore allow us to find the maximizing set for the graph and \( \tau_{\text{RB}}(H_1 \sqcup H_2) \).

One such set would cover part of one copy of \( H_2 \). However, note that if the root vertex of \( H_2 \) were included in \( F \), we would have to include all the descendants of \( H_2 \), since otherwise \( F \) would not be connected. Therefore, this class will only include subsets of \( H_2 \) which do not include the root vertex. In this case, we must maximize over all possible subsets of \( H_2 \) to find the best \( \tau(F) \). This may seem like it would yield \( \tau_{\text{RB}}(H_2) \); however, in this instance we can pick subsets of \( H_2 \) which make up a majority of \( H_2 \), which is not allowed for \( \tau_{\text{RB}} \). We define the un\- restricted rainbow time as

\[
\tau_{\text{RB}}(H_1 \sqcup H_2) = \max (u_{\text{RB}}(H_2), |H_2| \tau_{\text{RB}}(H_1)).
\] 

(S7)

We now seek to apply this to hierarchies \( G^{\Pi_{a^k}} \). Just as before, if a vertex is included in \( F \), we must also include in \( F \) all its descendants in the hierarchy, otherwise the complement \( \overline{F} \) will not be connected. Therefore, all bipartitions can be reduced to choosing a particular level of the hierarchy to cut — on that level, either a vertex will be included or not included, and this must apply to all of its descendants as well. Every bipartition can then be mapped to a bipartition of \( G \), but one where every vertex is scaled by \(|G|^{-1} \) due to the size of each sub-hierarchy [note that the large number of vertices not in \( F \) do not contribute to \( \tau(F) \)]. In addition, \( \tau(F) \) must also be modified by the edge weight, which we define to be \( \alpha_i \) on level \( i \).

There is one important difference between the top \((k)\) level and all others, which arises from the constraint that \(|F| \leq \frac{1}{2} |G^{\Pi_{a^k}}| \). A cut on the top level must not include more than half of the highest-level copy of \( G \), while all lower levels can use any cut at all as long as it does not include the root vertex. Whatever level we cut, the cut depends only on the base graph \( G \), with each node standing for \(|G|^{-1} \) total nodes below it. Therefore, we can write the overall \( \tau_{\text{RB}} \) as a maximization over these options:

\[
\tau_{\text{RB}}(G^{\Pi_{a^k}}) = \max \left( \frac{|G|^{k-1}}{a_k} \tau_{\text{RB}}(G), \sup_{i < k} \frac{|G|^{i-1}}{\alpha_i} u_{\text{RB}}(G) \right).
\] 

(S8)

For specificity, we will evaluate the case where \( G = K_n \), the complete graph, and \( \alpha_i = a^{i-1} \), which was proposed in Ref. [S5] as an architecture. Here, the maximization over lower levels [the second term in Eq. (S8)] can be reduced...
to either the first level or the $k - 1$ level, since we simply have to pick the largest element in a geometric sequence defined by $n/\alpha$. We can write the resulting maximization as a choice between three options,

$$
\tau_{RB}(K_n^{\Pi_\alpha k}) = \max \left( 1, \left( \frac{n}{\alpha} \right)^{k-1} \frac{2}{\alpha}, \left( \frac{n}{\alpha} \right)^{k-2} \right). \tag{S9}
$$

Whereas one might have expected two options to arise (cut at the top or at the bottom), we actually have three. For $\alpha > n$, the edges grow in capacity too quickly for the increased volume to make a higher-level cut worthwhile, so the optimal cut is at the bottom, yielding a constant scaling with $n$. Two other options appear at $n > \alpha$, where cutting higher up the hierarchy allows for greater volume of qubits in $F$ without too much penalty caused by changing edge weights. The reason there are two strategies is that it may be possible to cut a larger portion of a lower hierarchy and exploit the split between $\tau_{RB}$ and $u_{RB}$. (For $K_n$ in particular, the cut that includes all but the root vertex satisfies $u_{RB}(K_n)$.)

<table>
<thead>
<tr>
<th>Graph Name</th>
<th>$\tau_{RB}$</th>
<th>$w$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_N$</td>
<td>$N^{-1}$</td>
<td>$N^2$</td>
<td>$N$</td>
</tr>
<tr>
<td>$S_N$</td>
<td>$1$</td>
<td>$N$</td>
<td>$N$</td>
</tr>
<tr>
<td>$d$-dimensional Grid</td>
<td>$N^{1/d}$</td>
<td>$N$</td>
<td>$2d$</td>
</tr>
<tr>
<td>$K_n^{\Pi_\alpha k}$</td>
<td>$N^{\max(0, \log_\alpha \alpha)} \cdot N^{\max(1, \log_\alpha \alpha)} \cdot \log_\alpha N$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE I. Important statistics for graphs. Here, only the asymptotic scaling with $N$ is written. In addition to the rainbow time $\tau_{RB}$ for each graph, we also include the total weight of all edges $w$, and the maximum graph degree $\Delta$. Rainbow times for graphs other than hierarchies can be found in terms of isoperimetric number in Refs. [S6, S7].

To place these results in context, we compare the rainbow time of $K_n^{\Pi_\alpha k}$ to the total rainbow time of other graphs. To do this, we write the rainbow time in terms of the total number of qubits in a graph, $N$, and concern ourself with the overall scaling. In this language, $\tau_{RB}(K_n^{\Pi_\alpha k}) = \Theta \left( N^{\max(0, \log_\alpha \alpha)} \right)$. We compare this to the rainbow time of some other graphs in Table I. References [S6, S7] give the isoperimetric number for $K_N$, $S_N$ (the star graph of $N$ nodes), and grids (which are Cartesian products of paths). Satisfying sets for these graphs are: for $K_N$ and $S_N$, an arbitrary half of the nodes; for grids, a hypercube placed in one corner that takes up half the total volume.

One goal would be to identify a set of parameters where a hierarchy outperforms a $d$-dimensional grid architecture. We are most concerned with comparing to the $d$-dimensional grid because the other candidates we present, $K_N$ and $S_N$, both have very large degree, making them impractical for scalable architectures, although both have been used for small quantum devices [S8]. We find that the rainbow time of the hierarchy with base graph $K_n$ and scaling constant $\alpha$ will be better (smaller) than that of the grid if $\alpha > n^{(d-1)/d}$. If it also holds that $n > \alpha$, then the hierarchy will accomplish this while also having lower total edge weight, i.e., less overall connection overhead. We conclude that a hierarchy $K_n^{\Pi_\alpha k}$ with $\alpha \in \left[ n^{1-1/d}, n \right]$ has both lower rainbow time and less total edge weight than a $d$-dimensional grid of qubits.

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