Non-equilibrium fixed points of coupled Ising models

Jeremy T. Young,1,* Alexey V. Gorshkov,1,2 Michael Foss-Feig,3 and Mohammad F. Maghrebi4

1Joint Quantum Institute, NIST/University of Maryland, College Park, Maryland 20742 USA
2Joint Center for Quantum Information and Computer Science, NIST/University of Maryland, College Park, Maryland 20742 USA
3United States Army Research Laboratory, Adelphi, Maryland 20783, USA
4Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48824 USA

(Dated: March 8, 2019)

Driven-dissipative systems can exhibit non-equilibrium phenomena that are absent in their equilibrium counterparts. However, phase transitions present in these systems generically exhibit an effectively classical equilibrium behavior in spite of their quantum non-equilibrium origin. In this paper, we show that multicritical points in driven-dissipative systems can give rise to genuinely non-equilibrium behavior. We investigate a non-equilibrium driven-dissipative model of interacting bosons that exhibits two distinct phase transitions: one from a high- to a low-density phase—reminiscent of a liquid-gas transition—and another to an antiferromagnetic phase. Each phase transition is described by the Ising universality class characterized by an (emergent or microscopic) \(Z_2\) symmetry. They, however, coalesce at a multicritical point giving rise to a non-equilibrium model of coupled Ising-like order parameters described by a \(Z_2 \times Z_2\) symmetry. Using a dynamical renormalization-group approach, we show that a pair of non-equilibrium fixed points (NEFPs) emerge that govern the long-distance critical behavior of the system. We elucidate various exotic features of these NEFPs. In particular, we show that a generic continuous scale invariance at criticality is reduced to a discrete scale invariance. This further results in complex-valued critical exponents, spiraling phase boundaries, and a complex Liouvillian gap even close to the phase transition. As direct evidence of the non-equilibrium nature of the NEFPs, we show that the fluctuation-dissipation relation is violated at all scales, leading to an effective temperature that becomes “hotter” and “hotter” at longer and longer wavelengths. Finally, we argue that this non-equilibrium behavior can be observed in cavity arrays with cross-Kerr nonlinearities.

I. INTRODUCTION

The increasing control over atomic, molecular, and optical (AMO) systems has provided new avenues into studying many-body physics that are not accessible in conventional condensed matter systems. In particular, driven-dissipative systems, defined by the competition between a coherent drive and dissipation due to the coupling to the environment, have emerged as a versatile setting to investigate non-equilibrium physics [1]. They are very naturally realized by a variety of emerging AMO quantum simulation platforms ranging from exciton-polariton fluids [2–6], to trapped ions [7, 8], to Rydberg gases [9–11], to circuit-QED platforms [12]. At long times, these systems approach a non-equilibrium steady state due to the interplay of drive and dissipation. The steady states can potentially harbor novel phases and exhibit exotic dynamics. Situated far from equilibrium, understanding the properties of these steady states requires methods beyond those suitable in or near equilibrium. The quest to realize and characterize macroscopic phases of these non-equilibrium systems has sparked a flurry of theoretical and experimental investigations.

Given their non-equilibrium dynamics, driven-dissipative systems are expected to exhibit universal, critical properties different from their equilibrium counterparts. In spite of this, it has become increasingly clear that a large class of driven-dissipative phase transitions fall under classical equilibrium universality classes. In particular, the equilibrium Ising universality class and more generally the model A dynamics of the Hohenberg-Halperin classification—describing the critical behavior of a non-conserved order parameter in or near equilibrium—have emerged in a variety of driven-dissipative phase transitions; these include bosonic/photonic Bose-Hubbard systems [13–16], various driven-dissipative spin models [15, 17–24] near an Ising [23–25], antiferromagnetic [26], or a limit-cycle phase [22] as well as driven-dissipative condensates consisting of polaritons [27, 28]. A possible exception is a two-dimensional driven-dissipative condensate where it has been argued that the non-equilibrium Kardar-Parisi-Zhang (KPZ) universality class governs the long-wavelength dynamics [29, 30]. Even in the latter case, experiments are consistent with an effective thermal behavior [31, 32], although the KPZ physics may emerge in larger systems not accessible experimentally. Moreover, it has proved particularly difficult to identify non-equilibrium universal behavior which is genuinely of a quantum nature as it requires extra fine-tuning [33] and, in some cases, coupling to non-Markovian baths too [34–36]. In general, an important goal of the field is to identify whether generic driven-dissipative systems can escape the pull of an effective equilibrium behavior and instead give rise to new non-equilibrium universality classes.

* Corresponding author: jtttyy27@umd.edu
An effective equilibrium behavior is not special to driven-dissipative quantum systems. In driven-diffusive classical systems too, where the drive as well as the dynamics are entirely classical, effective equilibrium seems to be remarkably robust. For instance, an Ising-type dynamics governing a non-conserved order parameter is proved to be stable against all dynamical, non-equilibrium perturbations [37]. More generally, the universal dynamics of various models in the Hohenberg-Halperin classification [38] are shown to be robust against non-equilibrium perturbations which violate detailed balance [39–46]; truly non-equilibrium behavior emerges under more constrained dynamics, for example, in the presence of a conserved order parameter in an anisotropic medium [47, 48]. In much of the previous work, situations have been considered where the phase transition is governed by a single order parameter. Due to the restriction that this places on the dynamics, a description based on an effective free energy often becomes available, hence the emergence of an effective equilibrium behavior.

In this work, we consider a driven-dissipative model that gives rise to multicular points defined by the joint transition of two order parameters. In particular, we investigate the interplay of two phase transitions, each of which has been studied extensively in driven-dissipative settings: One is the many-body analog to optical bistability, and in the other a sublattice symmetry is spontaneously broken leading to an antiferromagnetic ordering. A schematic illustration of this combination is shown in Fig. 1. With two order parameters, an immediate identification of an effective free energy is no longer possible, so one might hope that an effective equilibrium behavior can be evaded. Remarkably, we show that a new, genuinely non-equilibrium universal behavior emerges at the multicritical point, giving rise to exotic critical behavior and dynamics. In contrast to other proposals for non-equilibrium critical behavior based on fine-tuned non-local or non-Markovian dissipation [33–36], our approach relies on tuning the system parameters (such as drive and detuning, which are easy to control) to a multicritical point. In fact, the driven-dissipative setting of our model can be experimentally realized using the cross-Kerr nonlinearities in cavity arrays [18, 19].

In order to determine the critical behavior, we will employ the Keldysh-Schwinger and functional integral formalism suited for the non-equilibrium setting of driven-dissipative systems [15, 24, 27–30, 33–35, 49–52]. While the presence of two order parameters prevents an immediate free-energy description, the long-wavelength universal behavior—and whether or not the macroscopic behavior escapes an equilibrium fixed point (EFP)—is determined by investigating how the parameters evolve under a dynamical version of renormalization-group (RG) techniques [53].

The remainder of this paper is organized as follows. In section III, we discuss the phase diagram of the model and identify the multicritical points where two distinct phase transitions meet. In section IV, we present the RG analysis and show that a pair of new non-equilibrium fixed points (NEFPs) emerge that exhibit a variety of novel critical behaviors. In section V, we discuss an experimental setting based on cavity arrays to realize the multicritical points of our model. Finally, in section VI, we conclude our paper with a discussion of possible future directions which are motivated by the results of our work. In the Appendices, we present technical details omitted from the main text.

II. SUMMARY OF MAIN RESULTS

In this section, we present a summary of the main results of this paper. We consider a driven-dissipative model which displays two distinct phase transitions, each of which arise generically in various settings. The first one is a many-body version of bistability where two stable solutions arise with a low or high population of photons (or excitation of spins). In the thermodynamic limit, the bistable region is reduced to a line of first-order phase transitions that terminates at a critical point, reminiscent of a liquid-gas phase transition. The second type of phase transition is one to an antiferromagnetic phase where the population takes different values on the two sublattices (a/b) of the system. We shall consider a model where
these phase transitions coalesce at a multicritical point and investigate the exotic dynamics due to the interplay of the respective order parameters. These features are provided, for example, in a driven-dissipative model of weakly interacting bosons with nearest-neighbor density-density interactions on a d-dimensional cubic lattice. The coherent dynamics of the model is governed by the Hamiltonian

\[ H = \sum_i -\Delta a_i^\dagger a_i + \Omega(a_i^\dagger + a_i) + \sum_{\langle ij \rangle} -J(a_i^\dagger a_j + a_j^\dagger a_i) + V a_i^\dagger a_i a_j^\dagger a_j, \]  

where \( \Delta \) is the detuning of the drive, \( \Omega \) the drive strength, \( J \) the hopping strength, and \( V \) the strength of the nearest-neighbor interactions. The incoherent dynamics is due to loss of bosons, characterized by the Lindblad operators \( L_i = \sqrt{\Gamma} a_i \), where \( \Gamma \) defines the loss rate. The (mixed) state of the system \( \rho \) evolves under the quantum master equation

\[ \dot{\rho} = -i[H, \rho] + \sum_i L_i^\dagger L_i - \frac{1}{2} \{ \rho, L_i^\dagger L_i \}, \]

until it approaches a non-equilibrium steady state at long times where \( \dot{\rho} = 0 \). The interplay of the coherente drive (the linear term in the Hamiltonian) and dissipation together with the interaction tends to give rise to bistability, while the nearest-neighbor interaction can lead to an antiferromagnetic phase. We stress that our general results should hold beyond the specific model considered here; for example, the addition of on-site interactions or density-dependent hopping terms to our model also gives rise to multicritical points whose universal properties should be independent of the microscopic model considered. More generally, the relevant features of our bosonic model also arise in a variety of driven-dissipative systems including spin models \([15, 17–19, 22–24]\).

Each phase transition in our model is characterized by an Ising-like order parameter (low/high density in the antiferromagnetic transition). The simple structure of the order parameter, together with the incoherent nature of the dynamics, puts a strong constraint on the universal properties of the phase transition. Thus, it may be expected that each phase transition alone is described by the Ising universality class that also governs the Ising-type transitions in equilibrium. It can be argued, on more formal grounds, that this is indeed the case. Associating the order parameter \( \phi_1 \) with bistability and \( \phi_2 \) with antiferromagnetic ordering, their long-wavelength properties in the steady state are governed by a thermal distribution but with respect to the effective (classical) Hamiltonians

\[ H_1 = \int_x \frac{D_1}{2} |\nabla \phi_1|^2 + \frac{u_1}{4} \phi_1^4 + \frac{r_1}{2} \phi_1^2 + h \phi_1, \]  

\[ H_2 = \int_x \frac{D_2}{2} |\nabla \phi_2|^2 + \frac{u_2}{4} \phi_2^4 + \frac{r_2}{2} \phi_2^2, \]  

with \( D_i \) characterizing the stiffness, \( u_i \) the interaction strength, \( r_i \) the distance from the critical point, and \( h \) an effective magnetic field. Note that due to sublattice symmetry, there is no magnetic field associated with the antiferromagnetic phase. Furthermore, the incoherent nature of the model leads to stochastic Langevin-type dynamics of the order parameters as \([53]\)

\[ \gamma_i \partial_t \phi_i = -\frac{\delta H_i}{\delta \phi_i} + \xi_i, \]

where \( \gamma_i \) is a “friction” coefficient and \( \xi_i \) describes Gaussian white noise with correlations

\[ \langle \xi_i(t, x) \xi_j(t', x') \rangle = 2 \gamma_i T_i \delta(t - t') \delta(x - x'), \]

with \( T_i \) the effective temperature of the system. Near equilibrium, the “friction” coefficients \( \gamma_i \) control the rate at which the system relaxes to a thermal state via dissipating energy and thus is a purely dynamical quantity. The noise itself is related to the dissipation (or, friction) through temperature in what is known as the Einstein relation, which itself is a consequence of the fluctuation-dissipation theorem \([53]\).

The situation is entirely different in the vicinity of multicritical points where the two order parameters are generally coupled. Given the underlying symmetries, the dynamics can always be brought to the form

\[ \gamma_1 \partial_t \phi_1 = -\frac{\delta H_1}{\delta \phi_1} - g_{12} \phi_1 \phi_2^2 + \xi_1, \]  

\[ \gamma_2 \partial_t \phi_2 = -\frac{\delta H_2}{\delta \phi_2} - g_{21} \phi_2 \phi_1^2 + \xi_2. \]

Notice that the new terms that couple the two fields respect the underlying Ising symmetry of both order parameters. The noise correlations are given by Eq. (5); we shall again exploit our freedom in scaling the fields to set \( T_1 = T_2 = 1 \). With the two order parameters coupled, the condition for an effective equilibrium description becomes much more restrictive. A thermal description requires the entire dynamics to be described by a single Hamiltonian. This only occurs when \( g_{12} = g_{21} \), leading to the effective Hamiltonian

\[ \mathcal{H} = H_1 + H_2 + \frac{g_{12}}{2} \int_x \phi_1^2 \phi_2^2, \]
FIG. 2. A schematic RG flow diagram projected to the $g_{12}$-$g_{21}$ plane. (The full RG flow requires a 5-dimensional space; see Sec. IV.) In addition to the EFPs where $g_{12} = g_{21}$ (green circles), a pair of stable NEFPs (orange diamonds) emerge in the sector defined by the opposite signs of $g_{12}$ and $g_{21}$. These new fixed points exhibit exotic critical behavior reflecting their truly non-equilibrium nature. Filled (black) arrows represent the stability while partial (gray) arrows indicate the expected stability of the various fixed points in different directions. Stability is known to lowest order in $\varepsilon$ expected stability of the various fixed points in different directions. Stability is known to lowest order in $\varepsilon$.

A. Scaling phenomena

In the vicinity of an RG fixed point governing a phase transition, the system exhibits universal scaling behavior characterized by critical exponents, regardless of the microscopic model. The scaling behavior of the correlation and response functions at a NEFP or in its vicinity takes, respectively, the form

$$C(\mathbf{q}, \omega) \equiv \mathcal{F}(\{a^\dagger(\mathbf{x}, t), a(0, 0)\}_c) \propto |\mathbf{q}|^{-2+\gamma - \nu z} \tilde{C} \left( \frac{\omega}{|\mathbf{q}|^z}, \frac{r}{|\mathbf{q}|^{1/\nu'}}, P \left( \frac{\log |\mathbf{q}|}{\nu''} \right) \right),$$

$$\chi(\mathbf{q}, \omega) \equiv i\mathcal{F}\Theta(t)(\{a^\dagger(\mathbf{x}, t), a(0, 0)\}) \propto |\mathbf{q}|^{-2+\eta - \nu' z} \tilde{\chi} \left( \frac{\omega}{|\mathbf{q}|^z}, \frac{r}{|\mathbf{q}|^{1/\nu'}}, P \left( \frac{\log |\mathbf{q}|}{\nu''} \right) \right),$$

where $\mathcal{F}$ denotes the Fourier transform in both space ($\mathbf{x}$) and time ($t$) with $\mathbf{q}$ the momentum and $\omega$ the frequency, the curly brackets denote the anti-commutator, and the subscript $c$ indicates the connected part of the correlation function. Here, $r = \sqrt{\nu_{12}^2 + \nu_{21}^2}$ is the distance from the multicritical point, $P$ is a $2\pi$ periodic function, and the functions $C$ and $\chi$ are dimensionless scaling functions. While in principle the scaling behavior could be different for the two order parameters ($\phi_1$ and $\phi_2$), we shall argue, based on a systematic RG analysis, that a stronger notion of scaling emerges where the critical (static and dynamic) behavior and exponents characterizing the two order parameters become identical. This is why we can express either the correlation or the response function via a single scaling function (and not one for each order parameter) with the same set of exponents. The exponents $\eta$ and $\eta'$ define the anomalous dimensions corresponding to correlation and response functions, respectively, and $z$ is the dynamical critical exponent characterizing the relative scaling of time with respect to spatial coordinates. The correlation length $\xi$ is described by the critical exponent $\nu'$ via $\xi \propto r^{\nu' - 1}$. Typically, it is the exponent $\nu$ associated with the scaling behavior of $r_1$ and $r_2$ that describes the scaling of the correlation length. However, the latter exponent becomes complex-valued at the NEFPs, $\nu^{-1} = \nu'^{-1} + i\nu''^{-1}$, with the real part determining the scaling of correlation function and the imaginary part leading to a discrete scale invariance, as we shall discuss shortly. Altogether, there are five independent critical exponents of interest: $\nu', \nu'', \eta, \eta', z$.

Critical points are generically associated with a continuous scale invariance where the system becomes self-similar upon an arbitrary rescaling of the momentum and frequency. However, due to the “log-periodic” function in the scaling functions, the correlations are self similar in which case the steady-state distribution is given by $e^{-\mathcal{H}}$. However, this will not generally be the case, so we must consider how various parameters flow under RG. While the microscopic (though coarse-grained) dynamics is not immediately described by a thermal state, it could be very well the case that the RG flow pulls the system into a thermal fixed point where $\mathcal{H}$ could be very well the case that the RG flow pulls the system into a thermal fixed point where $\mathcal{H}$
upon the rescaling $q \rightarrow b_\ast q$ and $\omega \rightarrow b_\ast^\eta \omega$ for a particular scaling factor
\[ b_\ast = e^{2\pi \nu'}, \] (9)
or any integer powers thereof. Rather than a typical continuous scale invariance, this behavior is indicative of a discrete scale invariance, reminiscent of fractals, shapes that are self similar under particular choices of scaling [54]. A schematic depiction of the correlation functions with discrete scale invariance is shown in Fig. 3(a). Similar examples of this behavior exist in earthquakes [55], stock markets [56], (equilibrium) systems on a fractal geometry [57], and via Efimov physics in quantum systems [58–61]. Additionally, since the origin of the discrete scale invariance is the scaling behavior of $r_\ast$ that characterize the distance from the critical point, the phase boundaries themselves also exhibit a form of discrete scale invariance in $r_\ast$; see Fig. 4 and the discussion in the next subsection titled “Phase diagram”.

The critical exponents $\eta$ and $\eta'$ characterize the anomalous dimensions corresponding to fluctuations and dissipation, respectively. In an equilibrium setting, the fluctuation-dissipation theorem dictates that the correlation and response functions are related as [53]
\[ C(q, \omega) = \frac{2T}{\omega} \text{Im} \chi(q, \omega). \] (10)
(We have assumed the classical limit of the fluctuation-dissipation theorem at low frequencies and at a finite temperature.) In an equilibrium setting, the temperature $T$ is just a constant set by an external bath and thus is scale invariant. Therefore, the overall scaling behavior of the correlation and response functions is identical apart from the dynamical scaling (due to $\omega^{-1}$ on the rhs of the above equation) set by the critical exponent $\eta$, so fluctuation-dissipation relations put a constraint on the critical exponents $\eta = \eta'$ for effectively equilibrium phase transitions. However, at the NEFPs, we obtain $\eta \neq \eta'$, indicating the violation of the fluctuation-dissipation theorem and resulting in a new exponent characterizing the non-equilibrium nature of the fixed point. This in turn results in an effective temperature that remains scale-dependent at all scales. Inspired by the fluctuation-dissipation theorem, we define an effective temperature as
\[ \text{Env}[C(q, \omega)] = \frac{2T'_{\text{eff}}(q, \omega)}{\omega} \text{Env}[\text{Im} \chi(q, \omega)]. \] (11)
To factor out the log-periodic nature of the correlations, we have made a convenient choice by postulating a fluctuation-dissipation relation between the envelope (Env) functions of the correlation and response functions. This relation can be defined via either the upper or lower envelope functions. We can then identify the scaling behavior of the effective temperature at the NEFP. Interestingly, the system gets “hotter” and “hotter” at longer and longer scales, characterized by an effective temperature that scales as $T'_{\text{eff}} \sim |q|^\eta-\eta'$ at long wavelengths and

<table>
<thead>
<tr>
<th>Non-equilibrium fixed points</th>
<th>Effective equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Scale invariance</td>
<td>Continuous scale invariance</td>
</tr>
<tr>
<td><img src="q" alt="Correlations" /></td>
<td><img src="q" alt="Correlations" /></td>
</tr>
<tr>
<td>(b) Effective temperature behavior</td>
<td><img src="q" alt="Temperature" /></td>
</tr>
<tr>
<td><img src="q" alt="Temperature" /></td>
<td><img src="q" alt="Temperature" /></td>
</tr>
<tr>
<td>(c) Liouvillian gap closure</td>
<td><img src="q" alt="Gap" /></td>
</tr>
<tr>
<td><img src="q" alt="Gap" /></td>
<td><img src="q" alt="Gap" /></td>
</tr>
<tr>
<td>(d) Critical exponents</td>
<td>$\nu^\ast = 2 - \left(\frac{1}{2} \pm \frac{1}{2 \sqrt{3}}\right) \epsilon$</td>
</tr>
<tr>
<td>$\eta = \frac{c}{\lambda_c}(1 - 12 \log(4/3)) &lt; 0$</td>
<td>$\eta &gt; 0$</td>
</tr>
<tr>
<td>$\eta' = \frac{c}{\lambda_c} \eta' = \eta$</td>
<td>$\eta' = \eta$</td>
</tr>
<tr>
<td>$z - 2 = \eta' (6 \log(4/3) - 1)$</td>
<td>$z - 2 = \eta' (6 \log(4/3) - 1)$</td>
</tr>
</tbody>
</table>

FIG. 3. Summary of the main features of the NEFPs contrasted with effective EFPs. (a) Schematic correlation functions. A generic continuous scale invariance characteristic of criticality is reduced to a discrete scale invariance at the NEFP. (b) Effective temperatures $T_\ast$ representing the two order parameters as a function of the length scale $(q^{-1})$. The two temperatures become identical at long length scales, but, while they approach a constant at the EFP, they diverge at large scales at the NEFPs. (c) Gap closure upon approaching the critical point. $\Gamma_e$ denotes the Liouvillian gap with the real part describing the relaxation rate (a.k.a. the dissipative gap) and the imaginary part characterizing the “coherence gap”. For the EFP, the gap can close only along the real line, indicated by the arrow. In contrast, the gap for the NEFP can take on complex values and close along any path lying in the shaded region, making a maximum angle of $\pi/3$ with the real line. (d) Critical exponents to lowest non-trivial order in $\epsilon = 4 - d$. The exponent $\nu$, typically associated with the divergence of the correlation length, becomes complex-valued at the NEFPs with its imaginary part characterizing the discrete scale invariance [cf. part (a)]. $\eta$ and $\eta'$ are anomalous dimensions characterizing fluctuations and dissipation with $\eta \neq \eta'$ at the NEFPs indicating the violation of the fluctuation-dissipation theorem. $z$ is the dynamical critical exponent.
FIG. 4. Phase diagram associated with the NEFPs of the non-equilibrium Ising model of two coupled fields for $h = 0$. Dis (white) indicates the disordered phase, B (red vertical shading) corresponds to the phase where the bistability order parameter undergoes spontaneous symmetry breaking, AF (blue horizontal shading) denotes antiferromagnetic ordering, and B+AF (purple square shading) corresponds to the phase where both order parameters are nonzero. The solid black lines denote second-order phase transitions. The NEFP phase diagram exhibits logarithmic spirals in the phase boundaries. The other NEFP is described by an analogous diagram upon switching the roles of the two order parameters ($B \leftrightarrow AF$, $r_1 \leftrightarrow r_2$).

fixed $\omega/|q|^z$. This behavior is illustrated in Fig. 3(b) individually for the two effective temperatures corresponding to the two order parameters. At long wavelengths, these effective temperatures become identical to each other and to $T^\text{eff}(q, \omega)$ defined above. Finally, the values of the critical exponents at the NEFPs are provided to the lowest non-trivial order in $\epsilon = 4 - d$ in Fig. 3(d).

B. Phase diagram

The critical point described by the new fixed points is a tetracritical point. In the vicinity of the tetracritical point (with $h = 0$), there are four different phases where none, one, or both order parameters undergo a continuous phase transition. A particularly exotic feature of the phase diagram is that it exhibits spiraling phase boundaries. This leads to the discrete scale invariance of the phase diagram itself, a property that follows from the same feature of the scaling functions in Eq. (8). In contrast, depending on the microscopic model, the EFPs can give rise to either a bicritical point—in which case there will not be a phase where both order parameters undergo a continuous phase transition—or a tetracritical point; neither of these will exhibit spiraling phase boundaries. Note that since the $\mathbb{Z}_2$ symmetry associated with the bistability transition ($\phi_1 \rightarrow -\phi_1$) is only an emergent one (when $h = 0$), the full phase diagram (including $h \neq 0$) can be better described as a three-dimensional plot that also includes the first-order phase transitions characteristic of bistability; see Fig. 7. The contrast between the EFPs and NEFPs can further provide a route to experimentally identify the new fixed points. An overview of the properties of bicritical and tetracritical points in equilibrium systems can be found in Refs. [62–68].

C. Spectral properties

The NEFP can be further distinguished by its particular dynamics that governs the relaxation of the system to the steady state. In the non-equilibrium setting of our model, the dynamics is described by the Liouvillian $\mathcal{L}$ via [cf. Eq. (2)]

$$\partial_t \rho = \mathcal{L}[\rho],$$

rather than a Hamiltonian. However, in analogy with the ground state that is described by the smallest eigenvalue of the Hamiltonian, the steady state(s) is given by the 0 eigenvalue(s) of the Liouvillian; all the other eigenvalues of the Liouvillian have a negative real part characterizing the decay into the steady state. Furthermore, the spectral gap of the Hamiltonian is naturally generalized to the eigenvalue of the Liouvillian with the smallest (in magnitude) nonzero real part. We denote this eigenvalue by $\Gamma_{\mathcal{L}}$. For a continuous phase transition, just like the spectral gap, the closing of the Liouvillian gap results in the divergence of a time scale associated with a slow or soft mode of the dynamics. The fashion that the latter gap closes reveals characteristic information about the nature of the phase transition. In equilibrium phase transitions at finite temperature, this gap becomes real (or, purely dissipative) as the critical point is approached. Even when the microscopic dynamics is far from equilibrium, the Liouvillian gap may (and typically does) become real, leading to effectively thermal equilibrium. In contrast, the dynamics near the NEFPs can close away from the real axis. This indeed occurs in the doubly-ordered phase; let $M_i = (\phi_i) \neq 0$ define the nonzero order parameters there and redefine the fields as $\phi_i \rightarrow \phi_i + M_i$. We then find the linearized equations of motion as

$$\gamma_1 \partial_t \phi_1 = 2g_1 M_1^2 \phi_1 - 2g_{12} M_1 M_2 \phi_2,$$

$$\gamma_2 \partial_t \phi_2 = 2g_2 M_2^2 \phi_2 - 2g_{21} M_1 M_2 \phi_1,$$

where, at the NEFPs, $g_{12} = -g_{21}$ and $\gamma_1^*/\gamma_2^* = 1$; noise, gradient, and higher-order terms have been dropped.
FIG. 5. Mean field dynamics near the NEFP within the doubly ordered phase with $|M_1| = |M_2|$. The arrows denote how the fields $\phi$ evolve in time, with four possible steady states. At each steady state, there is a dissipative relaxation process as well as a “coherent” rotation, resulting in a spiraling relaxation to the steady state. Two of the steady states spiral clockwise while two spiral counter-clockwise.

Due to the opposite signs of $g_{12}$ and $g_{21}$ in the two equations, we find a spiral relaxation to the steady state. This in turn is characterized by a complex Liouvillian gap—defined by a conjugate pair of complex eigenvalues—which exhibits both a dissipative (real) and a “coherent” (imaginary) part depending on the values of $M_1$ and $M_2$. We find that, when $|M_1| = |M_2|$, the angle of this complex gap relative to the real line achieves its maximum value of $\pi/3$. This is illustrated in Fig. 3(c). The corresponding mean-field relaxational dynamics is illustrated in Fig. 5.

III. MODEL

The representative model we have focused on is a driven-dissipative system of weakly interacting bosons defined in Eqs. (1,2). In order to understand how this model gives rise to bistability and antiferromagnetic ordering, we begin this section with a detailed discussion of mean field theory and corrections, or fluctuations, on top of the mean field solutions. Along the way, we will identify the soft modes of the dynamics that ultimately describe the critical behavior of the multicritical point. Finally, we conclude this section by presenting a mapping of our non-equilibrium model to a model of coupled Ising-like order parameters with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, corresponding to the sublattice symmetry as well as the emergent Ising symmetry due to bistability.

A. Mean field theory

In order to analyze the phase diagram of our model, we begin with a mean-field analysis, in which we assume different sites are uncorrelated, that is, for any two operators $A_i$ and $B_j$ on neighboring sites, we have $\langle A_i B_j \rangle = \langle A_i \rangle \langle B_j \rangle$ [69, 70]. Additionally, we assume that individual sites are described by coherent states. While the latter assumption follows from the former in our model, this will generally not be the case, for example, when on-site Hubbard interactions are present. Finally, in anticipation of the antiferromagnetic phase transition, we separate the system into two sublattices a/b and assume each to be described by a single coherent state.

Following these assumptions and using the fact that $\partial_t (\rho) = \text{Tr}(\rho O)$ for an arbitrary operator $O$, the resulting mean field equations of motion are given by

$$i \dot{\psi}_a = (-\Delta - i\Gamma/2) \psi_a - 3J |\psi_b|^2 \psi_a + \Omega,$$  

$$i \dot{\psi}_b = (-\Delta - i\Gamma/2) \psi_b - 3J |\psi_a|^2 \psi_b + \Omega,$$

where $\psi_i$ corresponds to the coherent state $\langle \psi \rangle$ on sublattice $i \in \{a, b\}$ and $z$ is the coordination number; from here on, we absorb $\zeta$ in the microscopic parameters via $3J \rightarrow J$ and $3V \rightarrow V$. It is clear from these equations that the density-density interaction behaves as an effective detuning that depends on the density of the other sublattice. This results in a similar physics as Rydberg excitations in stationary atoms, in which case the presence or absence of a Rydberg excitation on one site can either prevent (blockade) or facilitate (anti-blockade) a Rydberg excitation on a neighboring site by shifting it away or towards the effective resonance.

Setting $\psi_a = \psi_b$, one can immediately see that the mean-field equations become identical to those describing bistability; cf. Ref. [15] where the nonlinearity due to the Hubbard interaction should be replaced by the density-density interactions in this context. The emergence of bistability can be understood in simple terms: Away from resonance, there is a low population on each site. However, once a sufficient number of sites are highly excited, they begin to facilitate the excitation of neighboring sites, resulting in a high-population steady state. This process occurs when the shift in detuning due to interactions is comparable to the detuning. This condition is satisfied approximately when $\Omega^2 / (\Gamma^2 + (\Delta + J)^2) V \approx \Delta + J$, where $J$ behaves like an effective detuning while the product of the interaction strength $V$ and the non-interacting steady-state population $[\Omega^2 / (\Gamma^2 + (\Delta + J)^2)]$ gives the interaction-induced shift of the detuning. For $\Gamma \gtrsim \Delta + J$, this reasoning becomes blurred as the drive is effectively always on resonance due to the larger linewidth. As a result, a finite region of bistability emerges with low- and high-population steady states. Beyond mean field theory, the bistable region is replaced by a line of first-order phase transitions that terminates at a critical point.
The presence of antiferromagnetic ordering in this system can be understood by inspecting the role of the density-density interactions. Since the interaction affects neighboring sites only, the blockade effects occur between sublattices but not within each sublattice. For example, if one sublattice has a high population, it can prevent further excitations in the other sublattice. Similar to the case of bistability, the phase boundary occurs approximately when the shift in detuning due to interactions takes the system out of resonance. This approximately occurs when \( |\psi_A^2| V - \Delta - J | \geq \Gamma \), i.e., when one sublattice is effectively more than a linewidth out of resonance due to interactions. Unlike bistability, this process does not break down as \( \Gamma \) and \( \Omega \) are increased. As the decay \( \Gamma \) is increased, the drive strength \( \Omega \) can be further increased so that the interaction-induced shift in the detuning \( \phi \) takes the system out of resonance. This approximately when the shift in detuning due to interactions does not break down as \( \Gamma \) and \( \Omega \) are increased. As the density-density interactions. Since the interaction affects the critical values of \( \Delta + J \) we had not considered antiferromagnetic ordering. Thus, the critical values of \( \Delta + J \) of the equation of motion for \( \phi \) is increased, the drive strength \( \Omega \) can be further increased so that the interaction-induced shift in the detuning compensates for the increase of the linewidth.

In order to better understand the mean-field structure of the model, it is convenient to introduce a new set of fields corresponding to the two order parameters as

\[
\psi_B = \frac{\psi_a + \psi_b}{2}, \quad (15a)
\]

\[
\psi_{AF} = \frac{\psi_a - \psi_b}{2}. \quad (15b)
\]

The field \( \psi_B \) captures the effects of bistability while \( \psi_{AF} \) described the antiferromagnetic ordering. The mean-field equations can be in turn cast in terms of these fields as

\[i \dot{\psi}_B = (-\Delta - J - i \Gamma/2) \psi_B + V(\psi_B^2 - \psi_{AF}^2) \psi_B + \Omega, \quad (16a)\]

\[i \dot{\psi}_{AF} = (-\Delta + J - i \Gamma/2) \psi_{AF} + V(\psi_{AF}^2 - \psi_B^2) \psi_{AF}. \quad (16b)\]

At the multicritical point, \( \psi_{AF} = 0 \) and the equation governing the steady-state value of \( \psi_B \) is no different than if we had not considered antiferromagnetic ordering. Thus, the critical values of \( \Delta + J \), \( V \), \( \Omega \) as well as the steady-state value of \( \psi_B \) are determined according to the critical point associated with bistability only. This leaves a single free parameter in the equation of motion for \( \psi_{AF} \): \( \Delta - J \). By properly tuning the latter parameter, the antiferromagnetic phase boundary can be manipulated so that it intersects the critical point associated with bistability. Working in units of \( \Delta + J = 1 \), two multicritical points occur at

\[(\Delta_c, J_c) = \left( \frac{1}{3}, \frac{2}{3} \right) \quad \text{or} \quad \left( \frac{2}{3}, \frac{1}{3} \right), \quad (17)\]

and \( \Gamma_c = \sqrt{3/2} \), \( \Omega_c = (2/3)^{3/2}/\sqrt{V} \), as well as \( \Psi_c = \sqrt{2/3} V e^{-i\pi/3} \) as the steady-state value of \( \psi_B \) at the critical point (by virtue of symmetry, \( \psi_{AF} = 0 \) there).

The two fields \( \psi_{B/AF} \) are complex-valued, thus comprising four real (scalar) fields. However, given the Ising nature of each transition, we must anticipate that two scalar fields would be sufficient to describe the critical behavior of both types of ordering. Indeed we find that, at the multicritical point, two massless fields emerge—defined by appropriate components of the original fields—corresponding to the soft (or slow) modes, while the other components remain massive and are therefore noncritical (or fast). We then adiabatically eliminate the two noncritical modes by setting \( \phi_i = 0 \) and solving for \( \phi_i^* \) in terms of \( \phi_i \). Upon substituting our solutions for the massive fields into \( \phi_i \), we find an effective description in terms of the soft modes. We shall closely follow Refs. [15, 24] to identify these modes. For the bistability order parameter, we can identify

\[
\psi_B = \Psi_c e^{i\pi/3} (\phi_1 + \phi_1^*), \quad (18a)
\]

with the real fields \( \phi_1 \) and \( \phi_1^* \) characterizing the slow and fast modes, respectively. A similar identification has been made in Refs. [15, 24]; see also Refs. [71, 72] for a similar reasoning although the slow and fast modes identified there make a \( \pi/2 \) angle. For the antiferromagnetic field, the massless and massive components depend on the choice of the multicritical point in Eq. (17) as

\[
\Delta_c = 1/3 : \quad \psi_{AF} = \frac{1}{\sqrt{3}} \left( e^{-i\pi/6} \phi_2 + e^{i\pi/6} \phi_2^* \right), \quad (19a)
\]

\[
\Delta_c = 2/3 : \quad \psi_{AF} = \frac{1}{\sqrt{3}} \left( \phi_2 + e^{i\pi/3} \phi_2^* \right). \quad (19b)
\]

Again, the unprimed fields are massless while the primed fields are massive. The slow and fast modes of the fields are illustrated pictorially in Fig. 6.

Next we adiabatically eliminate the massive modes to find an effective description in terms of the soft modes. Including the gradient terms—describing the coupling between neighboring sites—as well as the noise terms due to the coupling to the environment, we find the Langevin equations

\[
\gamma_1 \dot{\phi}_1 = -h - r_1 \phi_1 + D_1 \nabla^2 \phi_1 + \xi_1 \\
+ A_{20} \phi_1^2 + A_{02} \phi_2^2 + A_{12} \phi_1 \phi_2^2 + A_{30} \phi_1^3, \quad (20a)
\]

\[
\gamma_2 \dot{\phi}_2 = -r_2 \phi_2 + D_2 \nabla^2 \phi_2 + \xi_2 \\
+ B_{11} \phi_1 \phi_2 + B_{21} \phi_1^2 \phi_2 + B_{03} \phi_2^3, \quad (20b)
\]

with Gaussian noise

\[
\langle \xi_i(t, \mathbf{x})\xi_j(t', \mathbf{x}') \rangle = 2\gamma_i T_i \delta_{ij} \delta(t - t') \delta(\mathbf{x} - \mathbf{x}'). \quad (21)
\]

Higher-order terms that are irrelevant in the sense of RG have been neglected. We have expressed the noise coefficients in a convenient notation that mimics the dissipative dynamics in thermal equilibrium, in spite of the underlying non-equilibrium dynamics. Finally, the details of the adiabatic elimination together with the explicit values of all the coefficients (\( h, r_s, D_s, A_s, B_s, \gamma_s \) etc.) have been presented in Refs. [15, 24].
φ

Shifting the field by a constant as
π/
ψ
respectively. (b) Relaxation of the field
FIG. 6. Dynamics of gapped (fast/massive) and soft (slow/massless) modes with arrows indicating the linear (in
symmetry:
free energy of a scalar field
φ
of redundant operators in an equilibrium setting.
the upper critical dimension; see Ref. [74] for a discussion
is an important step for perturbative RG and to identify
redundancy can be used to simplify our analysis. This
not change the long-distance behavior of the system, this
mations of the fields. Since such a transformation should
ated under suitably local symmetry-preserving transfor-
operators. These are terms in the action which are gener-
important to determine the upper critical dimension. In the
above example, the canonical scaling dimension of \( \phi \) is
given by \( \phi = \frac{d-2}{2} \). From this, the canonical scaling of an
interaction term can be determined; e.g., for a term of the
form \( g_n \int_\phi \phi^n \) in the free energy, we have \( g_n = d - n \frac{d-2}{2} \).
An interaction term becomes marginal when the corre-
sponding scaling dimension vanishes. Therefore, the
upper critical dimension determined by the \( \phi^n \) term is
\( d^{(n)}_c = \frac{n}{n-2} \). Had we naively started with the free en-
ergy in Eq. (22), the dimensional analysis would have
led us to conclude that the upper critical dimension is
six due to the cubic term. However, once we have taken
the redundant operator into account, the latter term van-
ishes while the quartic term determines the upper critical
dimension to be four.

Similar to the above example, we should first identify the redundant operators in the non-equilibrium setting
of the two coupled scalar fields \( \phi_1 \) and \( \phi_2 \). In this case,
we allow for a more general, nonlinear transformation
which is suitably local and retains the underlying sym-
metries. We find that the set of redundant operators in
and \( T \)s) in terms of microscopic parameters of the model
are provided in Appendix A.

It turns out that, at the level of mean field analysis,
\( A_{20} = 0 \) in the vicinity of the multicritical point. The
resulting mean-field dynamics (neglecting the gradient and
noise terms) of the two soft modes is then described by a
cusp-Hopf bifurcation; a detailed analysis of this type of
bifurcation can be found in Ref. [73]. However, since \( A_{20} \)
is not protected by any symmetries, the corresponding
term can be generated in the course of RG and become
of the order of the other quadratic terms. While we shall
focus on the multicritical points, further details about
the full mean field phase diagram of our model and slight
variations on it can be found in Refs. [18, 19].

B. Non-equilibrium Ising model for two fields

Before proceeding with our perturbative RG analysis,
it is important to identify what are known as redundant
operators. These are terms in the action which are gener-
ated under suitably local symmetry-preserving transfor-
ations of the fields. Since such a transformation should
not change the long-distance behavior of the system, this
redundancy can be used to simplify our analysis. This
is an important step for perturbative RG and to identify
the upper critical dimension; see Ref. [74] for a discussion
of redundant operators in an equilibrium setting.

As a simple illustrative example, consider the generic
free energy of a scalar field \( \phi \) in the absence of the \( \mathbb{Z}_2 \)
symmetry:

\[
F = \int d^4x \left[ (\nabla \phi)^2 + h\phi + r\phi^2 + u_3\phi^3 + u\phi^4 \right]. \tag{22}
\]

Shifting the field by a constant as \( \phi \to \phi + \phi_0 \), the free
energy is given by the same expression (up to an unim-
portant additive constant) with possibly different coef-
ficients. This underscores a redundancy in free energies
that describe the same physical system. The change of
the free energy \( \Delta F \) (or rather the integrand) due to a
constant shift of the field defines a redundant operator.
In particular, the cubic term transforms as

\[
u_3 \to u_3 + 4u\phi_0. \tag{23}\]

By choosing the value of \( \phi_0 \) properly, the \( \phi^3 \) term can be
dropped from the free energy while shifting the coef-
ficients of the terms \( \phi \) and \( \phi^2 \).

Identifying the redundant operator is particularly im-
portant to determine the upper critical dimension. In the
above example, the canonical scaling dimension of \( \phi \) is
given by \( \phi = \frac{d-2}{2} \). From this, the canonical scaling of an
interaction term can be determined; e.g., for a term of the
form \( g_n \int_\phi \phi^n \) in the free energy, we have \( g_n = d - n \frac{d-2}{2} \).
An interaction term becomes marginal when the corre-
sponding scaling dimension vanishes. Therefore, the
upper critical dimension determined by the \( \phi^n \) term is
\( d^{(n)}_c = \frac{n}{n-2} \). Had we naively started with the free en-
ergy in Eq. (22), the dimensional analysis would have
led us to conclude that the upper critical dimension is
six due to the cubic term. However, once we have taken
the redundant operator into account, the latter term van-
ishes while the quartic term determines the upper critical
dimension to be four.

Similar to the above example, we should first identify the redundant operators in the non-equilibrium setting
of the two coupled scalar fields \( \phi_1 \) and \( \phi_2 \). In this case,
we allow for a more general, nonlinear transformation
which is suitably local and retains the underlying sym-
metries. We find that the set of redundant operators in

\[
\Delta_c = 1/3
\]

\[
\Delta_c = 2/3
\]
our model is sufficient to remove all the quadratic terms
in the Langevin equation (or equivalently the cubic terms
in the action, similar to the free energy in the above ex-
ample); the details of this analysis are presented in Ap-
pendix B. In particular, we find that, under this trans-
formation, $A_{12}/B_{21} \to 2A_{02}/B_{11}$; therefore, the relative
sign of the quadratic terms (prior to the transformation)
determines the relative sign of the cubic terms in the fi-
nal equations of motion, a fact that will be important in
the subsequent analysis.

With the above considerations, the Langevin equations
can be finally brought into a canonical form as
\begin{align}
\gamma_1 \partial_t \phi_1 &= D_1 \nabla^2 \phi_1 - h_1 \phi_1 - g_1 \phi_1^3 - g_{12} \phi_1 \phi_2^2 + \xi_1, \\
\gamma_2 \partial_t \phi_2 &= D_2 \nabla^2 \phi_2 - r_2 \phi_2 - g_2 \phi_2^3 - g_{21} \phi_2 \phi_1^2 + \xi_2,
\end{align}
with Gaussian noise
\begin{align}
\langle \xi_i(t, x) \xi_j(t', x') \rangle = 2\gamma_i T_i \delta_{ij} \delta(t - t') \delta(x - x').
\end{align}
Therefore, the dynamics exhibits a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry
when $h = 0$, corresponding to the emergent symmetry
$\phi_1 \to -\phi_1$ in addition to the sublattice symmetry
$\phi_2 \to -\phi_2$. Such emergent symmetry has been previously
identified in the bistability transition [15, 23, 24]; our
analysis shows that that such symmetry emerges even in
the vicinity of a multicritical point where bistability and
antiferromagnetic transitions coalesce. We must point
out that, even in the absence of the latter symmetry, the
sublattice symmetry alone prevents any mixing of the
gradient and mass terms between the two fields, a prop-
erty that should hold to all orders of perturbation theory.

IV. RENORMALIZATION GROUP ANALYSIS

In this section, we derive the perturbative RG equa-
tions to the two-loop order (for the reasons that will be
explained shortly), identify the fixed points, and charac-
terize the critical exponents that determine the scaling
properties of correlations near the multicritical point.

A. RG equations

The Langevin-type equations can be cast in terms of
the Martin-Siggia-Rose-Janssen-De Dominicis functional
integral. This allows us to study our non-equilibrium
model by extending the standard techniques of the RG
analysis to a dynamical setting; see, for example, Ref. [53]
for more details. The non-equilibrium partition function
is defined by $Z = \int \mathcal{D}[\phi_i, \dot{\phi}_i] e^{-\mathcal{A}[\phi_i, \dot{\phi}_i]}$ where the func-
tional integral measure as well as the “action” $\mathcal{A}$ involve
both fields $\phi_i$ with $i = 1, 2$ and their corresponding “re-
response” fields $\tilde{\phi}_i$. In the language of Keldysh field the-
ory, $\phi$ corresponds to the classical field while $\tilde{\phi}/2i$ cor-
responds to the quantum field. The statistical weight
of $\phi_i(t, x)$ can be obtained by integrating out both re-
response fields as $P[\phi_i] = \int \mathcal{D}[\tilde{\phi}_i] e^{-\mathcal{A}[\tilde{\phi}_i, \phi_i]}$. While the
partition function $Z = 1$ by construction, the expec-
tation value of any quantity—the fields themselves or
their correlations—can be determined by computing a
weighted average in the partition function. For our model
defined by Eqs. (24.25), we write the action as the sum of
quadric and nonlinear (beyond quadratic) terms as
\begin{align}
\mathcal{A}[\tilde{\phi}_i, \phi_i] = \mathcal{A}_0[\tilde{\phi}_i, \phi_i] + \mathcal{A}_{\text{int}}[\tilde{\phi}_i, \phi_i],
\end{align}
with the quadratic action given by
\begin{align}
\mathcal{A}_0[\tilde{\phi}_i, \phi_i] &= \int_{t, x} h_1 \tilde{\phi}_1 + \sum_i \tilde{\phi}_i (\gamma_i \partial_t - D_i \nabla^2 + R_i) \phi_i - \gamma_i T_i \tilde{\phi}_i^2,
\end{align}
and the nonlinear interaction terms
\begin{align}
\mathcal{A}_{\text{int}}[\tilde{\phi}_i, \phi_i] &= \int_{t, x} g_1 \tilde{\phi}_1^3 \tilde{\phi}_1 + g_2 \tilde{\phi}_2^3 \tilde{\phi}_2 + g_{12} \phi_1 \phi_2^2 \phi_2 + g_{21} \phi_2 \phi_1^2 \phi_1.
\end{align}
Our goal is to determine the RG flow of various parame-
ters in the action and specifically of the coefficients $g$ of
the interaction terms.

We begin by considering the subspace defined by
$g_{12} g_{21} = 0$ when either $g_{12} = 0$ or $g_{21} = 0$. This
subspace is special in that it is closed under renormalization
to all orders. The reason is that, when $g_{12} = 0$ or $g_{21} = 0$, one
of the two fields is not affected by the other at the mi-
}croscopic level, a property that should hold at all scales.
This can also be understood perturbatively in a diag-
namatic scheme: If, say, $g_{21} = 0$ then all diagrams that
could generate $g_{12}$ involve a causality violation, hence it
should remain zero to all orders. An important conse-
fquence of this fact is that the relative sign of $g_{12}$ and
$g_{21}$ cannot change, as this would require passing through
the closed subspace.

Before performing the RG analysis, we first use our
freedom in rescaling the fields to cast the action in a more
convenient form. In Sec. II, we used this freedom to set
both temperatures to unity; here, for the convenience of
the RG analysis, we shall make a different choice. Note
that rescaling $\phi_2 \to c \phi_2$ and $\tilde{\phi}_2 \to \tilde{\phi}_2/c$ maps $g_2 \to c^2 g_2$,
$g_{12} \to c^2 g_{12}$, and $T_2 \to T_2/c^2$. Exploiting this freedom,
we can set the rescaled value of $g_{12}$ to be identical to $g_{21}$
up to a sign. In doing so, we have effectively shifted the
renormalization of $g_{12}/g_{21}$ onto $T_1/T_2$, simplifying
the RG analysis later. Note, however, since $g_{12}$ is rescaled
by a factor $c^2$, this transformation cannot change the
relative sign of $g_{12}$ and $g_{21}$. This is indeed consistent
with the closure of the $g_{12} g_{21} = 0$ subspace discussed
above. Having rescaled the fields appropriately, we can
write the action as (the quadratic action is repeated for
completeness)
\begin{align}
\mathcal{A}_0[\tilde{\phi}_i, \phi_i] &= \int_{t, x} h_1 \tilde{\phi}_1 + \sum_i \tilde{\phi}_i (\gamma_i \partial_t - D_i \nabla^2 + R_i) \phi_i - \gamma_i T_i \tilde{\phi}_i^2,
\end{align}
where \( \sigma = \pm 1 \) indicates the relative sign of \( g_{12} \) and \( g_{21} \) and the coefficients \( u_1 \), \( u_2 \), and \( u_{12} \) define the rescaled values of the interaction strengths (in an abuse of notation, we use the same notation for the other rescaled parameters of the model as well as the rescaled fields).

Let us first briefly consider \( \sigma = 1 \), in which case the action can be written in a suggestive form as

\[
\mathcal{A}[\tilde{\phi}_i, \phi_i] = \int_{t, x} \sum_i \tilde{\phi}_i \left( \gamma_i \partial_t \tilde{\phi}_i + \frac{\delta H}{\delta \tilde{\phi}_i} \right) - \gamma_i T_i \tilde{\phi}_i^2, \tag{28}\]

where the function \( H \) is given by

\[
H = \int_x \sum_{i=1,2} \left( \frac{D_i}{2} |\nabla \phi_i|^2 + \frac{\gamma_i}{2} \phi_i^2 + \frac{u_i}{4} \phi_i^4 \right) + h \phi_i + \frac{u_{12}}{2} \phi_1^2 \phi_2^2, \tag{29}\]

Put in this form, Eq. (28) bears close resemblance to an equilibrium setting where the dynamics is governed by a Hamiltonian (in this case, \( H \)). However, with each field at a different temperature, their coupled dynamics does not generally satisfy fluctuation-dissipation relations and thus an (effective) equilibrium behavior cannot be established at least at the microscopic level. (Note that unlike Sec. II, we have already used the scaling freedom in redefining the interaction parameters which in turn fixes the ratio \( T_1/T_2 \).) One then should resort to an RG analysis to determine whether or not effective equilibrium is restored at long wavelengths, that is, if \( T_1/T_2 \to 1 \) under RG. We shall see shortly that equilibrium proves to be a robust fixed point even when \( T_1/T_2 \neq 1 \) at the microscopic level.

In contrast, a Hamiltonian dynamics [similar to Eqs. (28,29)] is not possible when \( \sigma = -1 \) since a term proportional to \( \phi_1^2 \phi_2^2 \) in the Hamiltonian leads to equations of motion that couple the two fields with the same coefficient and hence the same sign. Therefore, in this case, the dynamics cannot flow to an EFP even when \( T_1 = T_2 \), with the exception of a decoupled fixed point where \( u_{12} = 0 \) (or \( g_{12} = g_{21} = 0 \)). Indeed we shall argue that a pair of genuinely non-equilibrium fixed points emerge in this case.

At a technical level, an RG analysis would be complicated as we need to consider diagrams up to two loops. This is because, at one loop, no renormalization occurs for the temperatures (due to causality) as well as the diffusion constants and friction terms (owing to their momentum and frequency dependence). This is while the interaction terms \( (u_1, u_2, \text{ and } u_{12}) \) are all renormalized already at one loop. This observation—beside aesthetic reasons—has motivated the representation adopted here; in the original description in terms of \( g_{12} \) and \( g_{21} \), the ratio \( g_{12}/g_{21} \) would not be normalized at one loop.

To perform the RG analysis, we first define the renormalized parameters as

\[
D_{ir} = Z_{D_i} D_i, \quad r_{ir} = Z_r r_i \mu^{-2},
\]

\[
u_{ir} = Z_u u_i \mu^{-\epsilon}, \quad u_{12r} = Z_{u_{12}} u_{12} \mu^{-\epsilon},
\]

\[
Y_{ir} = Z_{Y_i} Y_i, \quad T_{ir} = Z_T T_i,
\]

where \( A_d = 2^{1-d} \pi^{-d/2} \Gamma(3-d/2) \) is a geometrical factor, \( \Gamma(x) \) is Euler’s Gamma function, \( \mu \) is an arbitrary small momentum scale (compared to the lattice spacing), and \( \epsilon = 4-d \) defines the small parameter of the epsilon expansion. The effect of renormalization is captured in the \( Z \) factors that contain the divergences according to the minimal subtraction procedure. We shall determine these factors perturbatively to the lowest non-trivial order in \( \epsilon \) or loops (the details are provided in Appendix C). The lowest-order corrections to \( Z_r \) and \( Z_u \) occur at one loop (\( \sim \epsilon \)), while those of \( Z_y \), \( Z_T \), and \( Z_D \) appear at two loops (\( \sim \epsilon^2 \)). These perturbative corrections, while having some similarities with their equilibrium counterparts, are more complicated due to their non-equilibrium nature.

Using the above \( Z \) factors, we determine the RG flow and beta functions via

\[
\gamma_p = \mu \partial_\mu \ln(p_R/p), \tag{31a}\]

\[
\beta_{u_a} = \mu \partial_\mu u_{aR}, \tag{31b}\]

where \( p \in \{Y_i, D_i, T_i\} \) and \( u_a \in \{u_1, u_2, u_{12}\} \). These functions describe the flow of various parameters in the action under the change of the momentum scale \( \mu \). In particular, the beta functions identify the fixed points of the interaction coefficients via \( \beta_{u_a} = 0 \). At any such fixed point, the scaling behavior of the remaining parameters is governed by power laws whose exponents depend on \( \gamma_p \). Here, we report the beta functions for the interaction parameters \( u_a \) (the details are provided in Appendix C):

\[
\beta_{u_1} = u_{1R} \left( -\epsilon + 9 \frac{T_{1R}}{Y_{1R}^2 D_{1R}} u_{1R} \right) + \frac{3 T_{2R}}{Y_{2R}^2 D_{2R}} u_{12R}, \tag{32a}\]

\[
\beta_{u_2} = u_{2R} \left( -\epsilon + 9 \frac{T_{2R}}{Y_{2R}^2 D_{2R}} u_{2R} \right) + \frac{3 T_{1R}}{Y_{1R}^2 D_{1R}} u_{12R}, \tag{32b}\]

\[
\beta_{u_{12}} = u_{12R} \left( -\epsilon + 3 \frac{T_{1R}}{Y_{1R}^2 D_{1R}^2} u_{1R} + 3 \frac{T_{2R}}{Y_{2R}^2 D_{2R}^2} u_{2R} \right.
\]

\[
\quad + \frac{1}{Y_{1R} Y_{2R} D_{1R} D_{2R} (D_{1R} + D_{2R})} u_{12R} \right), \tag{32c}\]
where we have introduced $\tilde{D}_{iR} = D_{iR}/\gamma_{iR}$. These equa-
tions exhibit a number of important features. First, for $u_{12R} = 0$, we can absorb a factor of $T_{1R}/D_{1R}^2$ into $u_{1R}$, leaving the two beta functions for $u_i$ independent of
$T_i, D_i, \gamma_i$. We thus immediately recover a pair of uncoupled equi-
rium Ising phase transitions, as one should expect. Second, under equi-
librium conditions where $\sigma = 1$ and $T_{1R} = T_{2R} = T_R$, we recover the standard beta func-
tions in equilibrium. In a similar fashion, we can absorb the factors of $T_R/D_{1R}^2$ into $u_{iR}$ and $T_R/(D_{1R} D_{2R})$ into $u_{12R}$, again leaving the beta functions dependent only on the coupling coefficients. This observation underscores the
important fact that, in equilibrium, static properties are entirely decoupled from the dynamics. On the other hand, in the setting of our non-equilibrium model, statics and dynamics are inherently intertwined. Indeed, no re-
definition of the coupling terms can lead to beta functions that would be independent of $T_{1R}$ and $D_{1R}$. Things are different for $\gamma_{iR}$, as they can always be absorbed in other parameters; for example, we can still absorb $1/\gamma_{iR}$ into $u_{1R}$ and $1/(\gamma_{1R} \gamma_{2R})$ into $u_{12R}$ in the beta functions. This reflects the fact that, through an appropriate rescaling of the fields, one can always rescale $\gamma_i$ arbitrarily without changing $T_i, D_i$, or the overall structure of the action.

To set up the full RG equations, let us define the parameters
\begin{equation}
\nu = \frac{T_2}{T_1}, \quad w = \frac{D_2}{D_1},
\end{equation}
where we have defined $C = 9 \log(4/3) - 3/2$ and the functions
\begin{align}
F(w) &= -\frac{2}{w} \log \left( \frac{2 + 2w}{2 + w} \right), \\
G(w) &= \log \left( \frac{(1 + w)^2}{w(2 + w)} \right) - \frac{1}{2 + 3w + w^2}, \\
H(w) &= \frac{1}{w} \log \left( \frac{2 + 2w}{2 + w} \right) - \frac{3w + w^2}{8 + 12w + 4w^2}.
\end{align}

With these definitions, the beta functions for the new interaction parameters $\tilde{u}_i$ depend only on the renormalized parameters $\nu_R$ and $w_R$. To obtain the full RG equations, we further need to determine the RG evolution of the latter parameters. As we shall see, their RG equations are also closed in the (five) parameters defined in Eq. (33).

To see why, first notice that there are ten marginal parame-
ters in the original action at the upper critical dimension $(\mu, D_i, T_i, g_i, g_{12}/21)$ which can define the basin of attraction for the RG flow. Since all four fields and time can be rescaled relative to an overall momentum scale, this leaves a total of five parameters needed to de-
fine the fixed point. The remaining parameters $(r_i, h)$ define relevant directions of the RG flow and thus must be tuned to their critical values. In order to determine the RG equations for the parameters $v$ and $w$, we use the identity
\begin{equation}
\beta_{p/q} = \frac{p}{q} (\gamma_p - \gamma_q).
\end{equation}

We now report the full set of beta functions of the parame-
ters of our model (with $r_i$ and $h$ set to zero at the fixed point)
\begin{align}
\beta_{\tilde{u}_1} &= \tilde{u}_{1R} \left[ -\nu - 9\tilde{u}_{1R} \right] + \sigma v_R \tilde{u}_{12R}, \\
\beta_{\tilde{u}_2} &= \tilde{u}_{2R} \left[ -\nu - 9\tilde{u}_{2R} \right] + \sigma v_R \tilde{u}_{12R}, \\
\beta_{\tilde{u}_{12}} &= \tilde{u}_{12R} \left[ -\nu + 4\sigma v_R + w_R \tilde{u}_{12R} + 3\tilde{u}_{1R} + 3\tilde{u}_{2R} \right], \\
\beta_v &= -v_R \tilde{u}_{12R}^2 F(w_R) [v_R - \sigma][v_R + \sigma F(w_R^{-1})/F(w_R)], \\
\beta_w &= -w_R \left[ C \left( \tilde{u}_{1R}^2 - \tilde{u}_{2R}^2 \right) + \tilde{u}_{12R}^2 \left( v_R^2 G(w_R) - G(w_R^{-1}) \right) + 2\sigma v_R \tilde{u}_{12R} \left( H(w_R) - H(w_R^{-1}) \right) \right].
\end{align}

The functions $F, G, H$ always appear in the RG equa-
tions in pairs, with one taking $w_R$ and the other $w_R^{-1}$ as an argument. This is because the diagrams that con-
tribute to the beta functions come in pairs, correspond-
ing to one from the renormalization of the terms in-
volving $\phi_1$ only and the other from those that involve $\phi_2$ only. Similarly, under the mapping $\tilde{u}_{1R} \leftrightarrow \tilde{u}_{2R}$, $\tilde{u}_{12R} \rightarrow \sigma v_R \tilde{u}_{12R}, v_R \rightarrow v_R^{-1}$ and $w_R \rightarrow w_R^{-1}$, the beta functions are left unchanged. This reflects the fact that we can switch the role of $\phi_1$ and $\phi_2$ without changing
the physics. As a result, if either $\sigma = -1$ or $v_R \neq 1$ at a given fixed point, there will always be a second fixed point paired with it.

The above equations determine the full RG equations of our non-equilibrium model, but it is instructive to first consider the RG equations under equilibrium conditions where the temperatures are equal, i.e., $v_R = 1$, and $\sigma = 1$. We then immediately find that the temperature ratio does not flow, $\beta_v = 0$, hence the two temperatures remain identical at all scales. Furthermore, the temperature itself—and not just the ratio—remains scale invariant as $\gamma_T = 0$, indicating (effective) thermal equilibrium. Finally, as remarked earlier, the RG equations for the interaction terms become independent of $w_R$ under equilibrium conditions, highlighting once again the fact that, in equilibrium, the statics is decoupled from the dynamics.

There are two distinct scenarios with respect to the beta function $\beta_v$. The first scenario is that the beta function vanishes when $\gamma_D = \gamma_{D_2}$. Since the dynamical critical exponents are related to the flow of $D$ as $z_i = 2 + \gamma_D$, we find that $z_1 = z_2$ under this scenario. This means that both fields are governed by the same dynamical critical exponent, giving rise to a “strong dynamic scaling”. The second scenario occurs when $\gamma_D \neq \gamma_{D_2}$ which would lead to the fixed point $w_R = 0$ or $w_R = \infty$ depending on the sign of $\gamma_D - \gamma_{D_2}$. This behavior is then described by a “weak dynamic scaling” where the two fields exhibit different dynamical scaling properties and exponents [75–77]; see also [53]. Similarly, one can consider the beta function $\beta_w$ characterizing the RG flow of the ratio of the temperatures. In this case too, there are two scenarios: Either the beta function vanishes for a fixed temperature ratio or rather, depending on the sign of $\gamma_T_1 - \gamma_T_2$, the RG flow leads to either $v_R = 0$ or $v_R = \infty$, which both correspond to the $g_{12921} = 0$ subspace. However, this subspace does not appear to be amenable to perturbative RG and rather results in unphysical, divergent critical exponents. In this work, we shall restrict ourselves to the case where $v_R$ and $w_R$ are both finite and nonzero.

### B. Fixed points of RG flow

With the RG beta functions, we can now identify the resultant fixed points. In the $\sigma = 1$ sector, the only fixed points of the RG equations are those where $w_R^* = 1$, exhibiting a strong dynamic scaling, as well as $v_R^* = 1$, indicating that the two temperatures become identical at the fixed point. Indeed, aside from the case of $u_{12w} = 0$, the only possible fixed point value of $v_R$ at this order is 1. This can be seen by noting that the only other root of Eq. (35d) is $-F(\omega_R^{-1})/F(\omega_R)$, which is always negative and thus unphysical. Similarly, noting that the beta functions for $u_{12w}$ are identical at this order, all fixed points in this sector must satisfy $u_{12w} = u_{12w}^*$. In light of this, we immediately identify $w_R = 1$ as the only possible solution of Eq. (35e). Remarkably, an effective equilibrium behavior emerges in this sector despite the underlying non-equilibrium nature of the dynamics. In particular, we recover the familiar equilibrium $O(2)$ and biconical fixed points as well as various decoupled fixed points involving combinations of Gaussian and Ising fixed points. However, there are no additional NEFPs in this sector (possibly with the exception of a kind of weak dynamical scaling in the $g_{12921} = 0$ subspace). Note that the emergent equilibrium is not achieved by a simple rescaling of the terms in the action to mimic an effective free energy, but is truly the result of a nontrivial two-loop RG analysis.

In the $\sigma = -1$ sector, any nontrivial fixed point is truly non-equilibrium as it cannot be described by effective Hamiltonian dynamics that defines equilibrium. Therefore, we should first determine if there exists any nontrivial fixed point in this sector or alternatively if the RG evolution flows to a trivial (decoupled) fixed point. Interestingly enough, the former is the case; we find a pair of genuinely non-equilibrium fixed points as

$$v_R^* = 1, \quad w_R^* = 1,$$

$$u_{12w}^* = \epsilon, \quad u_{12w}^* = \epsilon, \quad u_{12w}^* = \pm \frac{\epsilon}{2\sqrt{3}}.$$

These fixed points too exhibit a strong dynamic scaling since $w_R^* = 1$, so the two fields are governed by the same dynamical scaling. Furthermore, we find $v_R^* = 1$, implying that the two temperatures are equal, which might suggest an equilibrium behavior; however, the latter temperatures only characterize the noise level while a true equilibrium description (and a genuine notion of temperature) requires Hamiltonian dynamics, which is inherently not possible in this sector.

While we have identified a new pair of NEFPs, this does not guarantee that they would govern the critical behavior near the multicritical point. If these fixed points are unstable under RG, further fine tuning would be necessary to access them. Even if they are stable, depending on the initial microscopic parameters, the system could still flow to an EFP under renormalization. Nevertheless we shall argue that the multicritical point is indeed governed by the new NEFPs.

To determine the stability of the fixed points, we need to consider the stability matrix

$$\Lambda_{ab} = \frac{\partial \beta_a}{\partial s_b R}, \quad (38)$$

where $s_b$ denotes the set of parameters that enter the RG equations. A fixed point is stable if all of the eigenvalues of $\Lambda$ are positive. Although we have determined the lowest order corrections to all five parameters, we can only determine these eigenvalues up to $O(\epsilon)$. This is because in order to fully determine $\Lambda$ to $O(\epsilon^2)$, we need to consider the two-loop corrections to the coupling terms $u$ and the three-loop corrections to $v, w$. To the order considered here, we find that three of the eigenvalues (corresponding to $u$) are positive and stable, while
the rest (corresponding to $v, w$) are 0 and thus marginal. We remark that the $O(2)$ EFP also exhibits the same stability to the same order. In equilibrium, however, one can also inspect the stability $w$ without having to go to higher-loop diagrams. This is because the the statics is decoupled from the dynamics at all orders, hence $A$ finds a block-triangular form where the static and dynamical sectors can be diagonalized separately. This makes it possible to inspect the stability of $w$ up to $O(\epsilon^2)$ at the same order of the RG calculations. Thus, in the $\sigma = -1$ ($\sigma = 1$) sector where their signs are different (the same), the system flows to the NEFP (EFP). While, in principle, non-perturbative effects or higher-order terms in $\epsilon$ could modify this behavior, this is a generic feature of perturbative RG and not specific to our model. A qualitative sketch of the expected RG flow is illustrated in Fig. 2 in terms of the original $g_{12}, g_{21}$ couplings.

Finally, we remark that in the case of the original microscopic model, $A_{02}$ and $B_{11}$ have opposite signs, which thus carries over to the the relative sign of $g_{12}$ and $g_{21}$. Thus it is plausible to expect a critical behavior governed by the NEFPs.

### C. Universal scaling behavior

Any fixed point—equilibrium or not—exhibits critical behavior and exponents characterizing correlations and dynamics among other properties of the system. In particular, we consider the anomalous dimensions $\eta$ and $\eta'$ of the original and response fields, the dynamical critical exponent $z$, and as well as the exponent $\nu$ characterizing the divergence of the correlation length with approaching the critical point. These exponents describe the scaling behavior of the correlation and response functions at or near criticality as

$$C_i(q, \omega, \{r_j\}) \propto q^{-2+\eta-z}C_i\left(\frac{\omega}{|q|^z}, \left\{\frac{r_j}{|q|^1/\nu}\right\}\right), \quad (39a)$$

$$\chi_i(q, \omega, \{r_j\}) \propto q^{-2+\eta'}\hat{\chi}_i\left(\frac{\omega}{|q|^z}, \left\{\frac{r_j}{|q|^1/\nu}\right\}\right), \quad (39b)$$

where $C_i, \hat{\chi}_i$ are general scaling (dimensionless) functions. We have dropped the subscript $i$ from $\eta, \eta', z$ due to the strong dynamic scaling and in anticipation of the same spatial scaling dimensions for the two fields; however, we have kept the subscript in $r_j$ for $j = 1, 2$ since the RG equations couple them in a nontrivial way.

The exponents at the fixed point can be extracted via what is known as the method of characteristics (see Appendix D for details). Noting that, for fixed bare (microscopic) parameters, the correlation and response functions are not affected by changing the RG momentum scale $\mu$, we can relate these critical exponents to the flow functions as

$$\eta = \gamma_T - \gamma_D, \quad \eta' = -\gamma_D, \quad z = 2 + \gamma_D - \gamma_Y. \quad (40)$$

The renormalization of the parameters $r_j$ and the corresponding exponent $\nu_j$ requires a special treatment and will be discussed later in this section. At the non-equilibrium critical point, we find [cf. Eqs. (31a,37) together with the $Z$ factors in Appendix C]

$$\gamma_Y = -\frac{\epsilon^2}{6} \log(4/3), \quad \gamma_D = -\frac{\epsilon^2}{36}, \quad \gamma_T = -\frac{\epsilon^2}{3} \log(4/3). \quad (41)$$

Interestingly, we see that, in contrast to an EFP where the temperature becomes scale-invariant, the effective temperature at the NEFPs changes with the scale. In particular, the system becomes “hotter” at longer length scales since $\gamma_T < 0$. Using Eq. (40), the critical exponents at the NEFPs are given by

$$\eta = \frac{\epsilon^2}{36} \left(1 - 12 \log \left(\frac{4}{3}\right)\right), \quad (42a)$$

$$\eta' = \frac{\epsilon^2}{36}. \quad (42b)$$

$$z = 2 + \eta' \left(6 \log \left(\frac{4}{3}\right) - 1\right). \quad (42c)$$

While, in equilibrium, $\eta = \eta'$ as a consequence of the fluctuation-dissipation theorem, we have $\eta \neq \eta'$ since the temperature itself is scale dependent, $\gamma_T \neq 0$, at the NEFP. Note also that the critical exponents $z, \eta, \eta'$ are the same for both fields. While strong dynamic scaling already guarantees the same dynamical critical exponent, the anomalous dimensions are also identical owing to the emergent symmetry of the fixed point where $u_{1R} = u_{2R}$ and $v_R = w_R = 1$. However, the latter do not reflect any actual symmetry of the model and could be modified at higher orders in epsilon expansion.

An interesting feature of the NEFPs is that $\eta < 0$ to the first nontrivial order in epsilon expansion. This is in contrast with equilibrium where $\eta > 0$, a fact that can be even proved on general grounds (e.g., unitarity in a related quantum field theory) [78]. If this feature ($\eta < 0$) extends beyond perturbation theory to, say, two dimensions, it would indicate that the correlation function ($C(r) \propto |r|^{-d+2-\eta}$) diverges at large distances. This would, however, invalidate the starting point of our field-theoretical treatment based on an expansion in field powers since large-scale fluctuations grow without bound. However, it might also indicate the absence of ordering in low dimensions. This possibility seems particularly natural in the light of the effective temperature increasing at larger scales, which in turn tends to disallow ordering in low dimensions. While this behavior may simply be an artifact of perturbative RG, it indicates that we should not expect the behavior of these NEFPs in low dimensions to be the same as for EFPs. Finally, we note that, to the lowest nontrivial order considered, the dynamical critical exponent $z$ is related to $\eta'$ in an identical fashion as in equilibrium.
Next we consider the renormalization of the mass terms. This requires special care since their renormalization is intertwined. Defining $\tilde{\mu}(l) = \mu l$ and the flowing parameters $\tilde{r}_1(l)$ with $\tilde{r}_1(1) = r_{1i}$, we find the flow equations \[ i\frac{d\tilde{r}_1(l)}{dl} = \gamma_{r_1} \tilde{r}_1(l) + \left(-2 + \frac{\epsilon}{2}\right) \tilde{r}_1(l) + \frac{\epsilon}{2\sqrt{3}} \tilde{r}_2(l), \] \[ i\frac{d\tilde{r}_2(l)}{dl} = \gamma_{r_2} \tilde{r}_2(l) + \left(-2 + \frac{\epsilon}{2}\right) \tilde{r}_2(l) + \frac{\epsilon}{2\sqrt{3}} \tilde{r}_1(l), \] where the $\pm$ refer to the two NEFPs with opposite signs of $\tilde{u}_{12i}$. The flow equations can be solved as \[ \tilde{r}_1(l) = l^{-1/\nu'} \left[r_{1i} \cos \frac{\log l}{\nu''} + r_{2i} \sin \frac{\log l}{\nu''}\right], \] \[ \tilde{r}_2(l) = l^{-1/\nu'} \left[r_{2i} \cos \frac{\log l}{\nu''} - r_{1i} \sin \frac{\log l}{\nu''}\right], \] where \[ \nu'^{-1} = 2 - \frac{\epsilon}{2}, \quad \nu''^{-1} = \pm \frac{\epsilon}{2\sqrt{3}}. \] These equations can be cast in a more compact notation as \[ \tilde{r}_1(l) + i\tilde{r}_2(l) = l^{-1/\nu'} \left[r_{1i} + ir_{2i}\right]. \] Put in this form, the critical exponent $\nu$ emerges as \[ \nu^{-1} = \nu'^{-1} \pm i \nu''^{-1} = 2 - \left(\frac{1}{2} \pm \frac{i}{2\sqrt{3}}\right) \epsilon. \] Interestingly, this exponent becomes complex-valued at the NEFP. We can then express the scaling functions in Eq. (39) as \[ \check{C}_i = \check{C}_{\epsilon} \left(\frac{\omega}{|q|^2}, \left|\frac{r_{1i}}{|q|^{1/\nu'}}\right|, P(1/\nu'' \log |q|)\right) \] \[ \check{x}_i = \check{x}_i \left(\frac{\omega}{|q|^2}, \left|\frac{r_{1i}}{|q|^{1/\nu'}}\right|, P(1/\nu'' \log |q|)\right) \] where \[ |r_{1i}| = \sqrt{r_{1i}^2 + r_{2i}^2} \] and $P$ is a $2\pi$-periodic function. To obtain these equations, we have used the fact any function of the form $r/l^{1/\nu' + 1/\nu''}$ can be instead written as function of $|r|/|q|^{1/\nu'} + e^{i(\log l)/\nu''}$. The former expression often appears in scaling functions of this type and characterizes the scaling of the correlation length; however, the latter gives rise to a log-periodic function as a change of $\log l \rightarrow \log l + 2\pi \nu''$ leaves the exponential invariant.

The appearance of log-periodic functions has important consequences for the critical nature of the fixed points. They lead to a discrete scale invariance rather than the characteristic continuous scale invariance at a typical critical point \[54\]. Rather than a self-similar behavior at all length scales, a preferred scaling factor emerges as \[ b_\epsilon = e^{2\pi \nu''}, \] rescaling by which, or any multiple integer thereof, leaves the system scale invariant. In this sense, discrete scale invariance mimicks a fractal-like structure, in which rescaling the system by a particular factor leaves the system self similar. Note however that the discrete scale invariance and the fractal-like structure only emerges at long length scales (in the continuum) as opposed to the microscopic structure of a fractal (in the discrete space). Additionally, if we were to consider, e.g., the effect of a physical momentum cutoff $\Lambda$, this would enter the periodic function as a phase shift, thus determining the phase of the oscillations.

Similar phenomena appear to arise in stock markets \[56\], earthquakes \[55\], equilibrium models on fractals \[57\] and several other systems \[54\]. Log-periodic functions and the emergence of a preferred scale have been identified in the early developments of renormalization group theory \[79–81\], but they have been rejected as artifacts of position-space RG. On the other hand, their recent surge in diverse contexts from earthquakes to stock markets has instead relied on simple dynamical systems (with one or few variables) where the dynamics involves a discrete map itself \[54\]. This phenomenon has also surged in recent works on the dynamics of strongly correlated systems \[82\] or in the context of non-equilibrium quantum criticality \[33\]. A particularly well-known example of RG limit cycles are the Efimov states \[60, 61\], whose binding energies form a geometric progression similar to discrete scale invariance. These quantum RG limit cycles have been noted to be closely related to Berezinskii-Kosterlitz-Thouless (BKT) phase transitions \[58, 59\]. Disordered systems provide another context where complex-valued exponents and discrete scale invariance have been noted in both classical \[83–87\] and quantum \[88\] settings. The discrete scale invariance reported in this work, however, appears to be unique as it has emerged in an effectively classical, yet non-equilibrium model in the absence of disorder.

The discrete scale invariance approaches a continuous one as the upper critical dimension, $d_c = 4$, is approached. In three dimensions, perturbative values at the NEFP (with $\epsilon = 1$) yield a very large scaling factor ($b_\epsilon \sim 10^9$); however, with the exponential dependence on the critical exponents, the scaling factor is sensitive even to small corrections of the exponent beyond the lowest-order perturbation theory. Nevertheless, our results should be viewed as a proof of principle for the emergence of discrete scale invariance in macroscopic non-equilibrium systems. Additionally, higher harmonics in the periodic function $P$ can be significant, which then should be observed over smaller variations in the physical
scale.

Finally, we elaborate on a possible connection between the log-periodic behavior and limit cycles. Indeed, the microscopic mean-field phase diagram near the multicritical point also includes a limit cycle phase that displays persistent oscillations. For a rapidly oscillating limit cycle, the corresponding phase transition can be described from the viewpoint of a rotating frame (defined by the oscillation frequency) and by making the rotating-wave approximation. With this mapping, a limit-cycle phase transition is no different from a dissipative phase transition with an emergent $U(1)$ symmetry [22]. Near our multicritical point, however, the frequency of oscillations becomes small and thus no such mapping is possible. On the other hand, the discrete scale invariance discussed above also leads to an oscillatory behavior (in both space and time), albeit one that is log-periodic. Nevertheless, a natural possibility is that, at some intermediate regime away from the multicritical point, the discrete scale invariance merges into a limit-cycle solution. Moreover, we shall see that, in the doubly-ordered phase, the Lifshitz gap becomes complex-valued (see Sec. IV E). This furthers the possible connection to the limit cycle phase as a nonzero imaginary part implies that the system undergoes oscillations—which however decay—as the steady state is approached.

In Table I, we summarize all of the fixed points (aside from those involving the trivial Gaussian fixed point) and their critical exponents to the lowest order.

<table>
<thead>
<tr>
<th>Fixed Point</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_{12}$</th>
<th>$\nu^{-1}$</th>
<th>$\eta$</th>
<th>$\eta'$</th>
<th>$z - 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NEFP</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{2} \pm \frac{3}{2} \sqrt{3}$</td>
<td>$2 - \frac{1}{2} \pm \frac{1}{2} \sqrt{3} \epsilon$</td>
<td>$\frac{\epsilon^2}{15}(1 - 12 \log \frac{\eta}{\epsilon})$</td>
<td>$\frac{\epsilon^2}{5}$</td>
<td>$\eta'(6 \log \frac{\eta}{\epsilon} - 1)$</td>
</tr>
<tr>
<td>$O(2)$</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{3}{10}$</td>
<td>$2 - \frac{2}{5} \epsilon$</td>
<td>$\frac{\epsilon^2}{20}$</td>
<td>$\frac{\epsilon^2}{14}$</td>
<td>$\frac{\epsilon^2}{54}$</td>
</tr>
<tr>
<td>Biconical</td>
<td>$\frac{3}{18}$</td>
<td>$\frac{3}{18}$</td>
<td>$\frac{5}{9}$</td>
<td>$2 - \frac{4}{9} \epsilon$</td>
<td>$\frac{\epsilon^2}{24}$</td>
<td>$\frac{\epsilon^2}{14}$</td>
<td>$\frac{\epsilon^2}{54}$</td>
</tr>
<tr>
<td>$Z_2 + Z_2$</td>
<td>$\frac{3}{7}$</td>
<td>$\frac{3}{9}$</td>
<td>0</td>
<td>$2 - \frac{4}{7} \epsilon$</td>
<td>$\frac{\epsilon^2}{24}$</td>
<td>$\frac{\epsilon^2}{14}$</td>
<td>$\frac{\epsilon^2}{54}$</td>
</tr>
</tbody>
</table>

**Table I.** Fixed point values of the coupling coefficients and critical exponents to the lowest order. In all cases, $v_R = w_R = 1$. At the NEFP, $\sigma = -1$. The decoupled $Z_2 + Z_2$ fixed point and the biconical fixed point in this case are unstable to the order $O(\epsilon)$, while the other two fixed points are stable to the same order. The $Z_2 + Z_2$ and biconical fixed points exhibit the same critical behavior since they can be mapped to each other through a $\pi/4$ rotation in the $\phi_1$-$\phi_2$ plane. Fixed points involving the Gaussian fixed point are not included.

D. Phase diagram

The phase diagram itself is distinct in the vicinity of the NEFPs. In contrast with their equilibrium counterparts, these fixed points give rise to a tetracritical point. With the effective magnetic field set to zero, $h = 0$, four different phases emerge: A disordered phase with $\phi_1 = \phi_2 = 0$; two phases with either $\phi_1 \neq 0$ corresponding to bistability or $\phi_2 \neq 0$ leading to antiferromagnetic ordering; and, finally, a doubly-ordered phase where both fields become ordered, $\phi_1 \neq 0 \neq \phi_2$. While the first three phases also emerge in the mean field theory of the microscopic model, the doubly-ordered phase only arises in the course of RG when the $A_{20}$ term is generated.

The phase boundaries are governed by the scaling behavior of $r_1$. Let us set the effective magnetic field to zero, $h = 0$, and consider the scaling functions characterizing the correlation and response functions in Eq. (39). To determine the phase boundary, it suffices to take the limit $\omega, q \to 0$. In this case, the scaling functions are solely determined as a $2\pi$-periodic function of $\frac{\epsilon}{\nu} \log \left( |r_R| - \angle r_R \right)$, where $\angle r_R$ is the polar angle in the $r_1$-$r_2$ plane; this is achieved by eliminating the momentum scale in Eq. (48) in favor of $r_R$. Since the correlation functions only depend on the mass terms through the above combination, the phase boundary itself—characterized by the divergence of correlations—arises at a fixed value of this quantity (up to integer multiples of $2\pi$). Therefore, the shape of the phase boundary is given by

$$\frac{\nu'}{\nu^2} \log(r_R) - \angle r_R = \text{const},$$

which is a spiral, leading to the phase diagram illustrated in Fig. 4. Similar to our discussion of the discrete scale invariance, the perturbative values of the exponents at $\epsilon = 1$ require very large scales to observe a full spiral. But again these scales are highly sensitive to the corrections to perturbative RG. Moreover, partial spirals can still be observed for reasonable scales. Since the spiraling boundaries all spiral in the same direction, distinguishing them from equilibrium critical points, the effects of this may be seen even for very weak spiraling. Additionally, the two NEFPs are distinguished from each other by the direction of spiraling, since each has a different sign of $\nu'$. When the effective magnetic field is nonzero, there is no such symmetry as $\phi_1 \to -\phi_1$, leading to a surface of first-order phase transitions where $\phi_1$ undergoes spontaneous symmetry breaking. This first-order transition occurs in both the uniform phase (defined by the $B$ phase at $h = 0$) and the antiferromagnetic phase (defined by the $B+AF$ phase at $h = 0$). In both cases, the average population changes discontinuously while the sublattice population difference changes continuously. Finally, we take into account the magnetic field in determining the phase boundaries by considering an effective mass as $r_R + |h_R|^{1/\beta}$ where $\beta$ and $1/\beta$ describe the scaling be-
havior of the order parameter with \( r \) and \( h \), respectively, within the ordered phase. The effect of the magnetic field \( h \) is to “unravel” the spirals for small \( r \). The corresponding phase diagram for nonzero effective magnetic field is illustrated in Fig. 7.

E. Liouvillian gap closure

In this section, we investigate how the Liouvillian gap closes upon approaching the multicritical point. In contrast to the EFPs where the gap always closes along the real axis (hence, purely dissipative or relaxational dynamics), the NEFPs exhibit a qualitatively different behavior with the gap closing along a complex path, distinguishing them from both typical classical and quantum equilibrium systems.

We consider the system in the doubly-ordered phase where both fields take nonzero expectation values \( M_i \); for notational convenience, we make the change of variable \( \phi_i \rightarrow \phi_i + M_i \) where the fields \( \phi_i \) now represent the fluctuations around the order parameter. In addition to the original action, this transformation introduces new quadratic and linear terms as (including the original \( r_1 \) and \( r_2 \) terms too)

\[
\int_{x,t} \left( r_1 + 3u_1M_i^2 + u_{12}M_i^2 \right) \phi_1 \dot{\phi}_1 + \left( r_2 + 3u_2 M_i^2 + \sigma u_{12} M_i^2 \right) \phi_2 \dot{\phi}_2 + 2u_{12}M_i M_2 \phi_2 \dot{\phi}_1 + 2\sigma u_{12} M_1 M_2 \phi_1 \dot{\phi}_2
\]

\[+ M_1 (r_1 + u_1 M_1^2 + u_{12} M_2^2) \dot{\phi}_1 + M_2 (r_2 + u_2 M_2^2 + \sigma u_{12} M_1^2) \dot{\phi}_2. \]  

(51)

In addition, several cubic terms are also introduced, which are not reported for simplicity. We set the vertices \( \dot{\phi}_1 \) and \( \dot{\phi}_2 \) to zero since, by definition, \( \phi_i \) solely represent the fluctuations. This in turns sets \( r_1 = -u_1M_i^2 - u_{12}M_2^2 \) and \( r_2 = -u_2M_2^2 - \sigma u_{12}M_1^2 \). The remaining quadratic vertices are then given by

\[ 2u_1 M_i^2 \phi_1 \ddot{\phi}_1 + 2u_2 M_2^2 \phi_2 \ddot{\phi}_2 + 2u_{12} M_1 M_2 (\phi_2 \ddot{\phi}_1 + \sigma \phi_1 \ddot{\phi}_2). \]

(52)

Note that this equation already includes the renormalization, with the \( u \) terms understood to assume their fixed-point values. Putting these terms together with the quadratic part of the action, we find

\[
S_0 = \int_{x,t} \sum_i \left( \gamma \dot{\phi}_i - D \nabla^2 + R_i \phi_i - \gamma T \dddot{\phi}_i + R_{12}(\phi_2 \ddot{\phi}_1 + \sigma \phi_1 \ddot{\phi}_2) \right) \]

where \( R_i = 2u_i M_i^2 \) and \( R_{12} = 2u_{12} M_1 M_2 \). The poles of

the corresponding propagators are then obtained as

\[
0 = \sigma R_{12}^2 - (DK^2 + R_1 + i\gamma \omega)(DK^2 + R_2 + i\gamma \omega),
\]

or more explicitly as

\[
-\gamma \omega = DK^2 + \frac{R_1 + R_2}{2} \pm \sqrt{\sigma R_{12}^2 + \left( \frac{R_1 - R_2}{2} \right)^2}. \]

(55)

From this equation, we can find a simple condition for when the poles do not lie along the imaginary axis (corresponding to the negative real eigenvalues of the Liouvillian) as

\[
-\sigma R_{12}^2 > \left( \frac{R_1 - R_2}{2} \right)^2.
\]

(56)

Indeed, in equilibrium, where \( \sigma = 1 \), this condition cannot be satisfied. This implies that the relaxation is purely
relaxational in equilibrium as expected (in this case, for model A). However, at the NEFPs where $\sigma = -1$, the above condition can be satisfied. To see this, let us cast the above condition for $\sigma = -1$ in terms of $u$ and $M_i$ as

$$4u_{12}^2 M_1^2 M_2^2 > (u_1 M_1^2 - u_2 M_2^2)^2. \quad (57)$$

Recalling that $u_1 = u_2$ at the NEFPs at least to the lowest order in epsilon expansion, the above condition is trivially satisfied whenever $|M_1| = |M_2|$. In this case, the pole with the lowest nonzero decay rate takes the form

$$- i \gamma \omega = 2 M^2 (u_1 \pm i u_{12}). \quad (58)$$

In fact, with $|M_1| = |M_2|$, the Liouvillian gap achieves its largest imaginary value relative to its real part. We can identify the ratio of the imaginary part to the real part of the gap as $\frac{u_{12}}{u_1} = \sqrt{3}$, which corresponds to the Liouvillian gap closing at the angle $\pi/3$ with respect to the real axis. A generalization of the model considered here to the $O(N) \times O(N)$ model of N-component vector-like order parameters leads to similar behavior. In fact, we find that, for $N = 2$ and $N = 3$, the corresponding Liouvillian gap closes at the angles $\pi/4$ and $\pi/6$, respectively. We should then conclude that different non-equilibrium universality classes of our model and its generalization give rise to different angles of the Liouvillian gap closure in the complex plane. Further details on these generalized models will be presented in follow-up papers.

The scaling of (the magnitude of) the gap itself as a function of the distance from the critical point can be directly obtained by that of $M_i^2 / \gamma$. The scaling of $M_i^2$ is characterized by the critical exponent $\beta$ via $M_i \propto |r|^\beta$ where $|r|$ denotes the distance from the critical point ($r$ and $M_i$ denote the bare, rather than the renormalized, values). This exponent is related to the previously-determined critical exponents via $\beta = \nu' (d - 2 + \eta)/2$; the exponent $\eta/2$ represents the corrections to the scaling of $\phi$ which in turn determines that of the order parameter $|r|$. With the order parameters $M_1$ and $M_2$ scaling similarly, the angle that defines the gap closure in the complex plane only depends on (the absolute value of) their ratio. As remarked earlier, this angle achieves its maximum when $|M_1| = |M_2|$. We further note that the gap is purely real (relaxational) near phase boundaries where only one of the order parameters undergoes a transition since the lhs of Eq. (57) would be suppressed compared to the rhs.

Finally, much like the discrete scale invariance in the previous section, the complex Liouvillian gap is somewhat reminiscent of limit cycles, although a true limit cycle phase is characterized by purely imaginary eigenvalues that characterize the steady state itself.

V. EXPERIMENTAL REALIZATION

An ideal avenue for realizing these multicritical points is via the use of cavity or circuit quantum electrodynamics (QED). Individual cavities and circuits have been studied experimentally in great depth due to their potential applications in quantum computation [12, 89, 90]. Furthermore, both cavity QED and circuit QED have been proposed as platforms for realizing many-body states of light via nearest-neighbor coupling arrays of cavities or circuits [91–94]. Generally, these cavities and circuits have non-negligible loss due to dissipation. While dissipation is detrimental when it comes to realizing the quantum ground state of a given system, it is a crucial ingredient in realizing driven-dissipative phase transitions. There has been a variety of theoretical proposals to realize different driven-dissipative models in cavity- and circuit-QED systems [18–20, 50, 95, 96]. Recent experiments have even identified a driven-dissipative many-body phase transition [97].

For the model considered in this work (see Sec. II), many-body experimental platforms already exist that include drive, hopping, as well as dissipation. The remaining ingredient is then the nearest-neighbor interaction (the quartic term in Eq. (1)) to be contrasted with a Hubbard term that characterizes on-site interaction. Both types of interaction are generally known as Kerr nonlinearities; we are interested in what is known as a cross-Kerr nonlinearity, which has been utilized experimentally in several few-mode systems [98–100]. A more general version of our model has been considered in Refs. [18, 19], along with a discussion on how the nonlinear interaction terms can be tuned experimentally via Josephson nanocircuits. In a recent theoretical proposal, a setting consisting of a capacitor in parallel with a superconducting quantum interference device (SQUID) is put forth as an alternative means of achieving tunable Kerr nonlinearities [101].

While generic experimental settings introduce other nonlinear terms (e.g., Hubbard interactions and correlated hopping) in addition to the density-density interactions, we do not expect them to dramatically affect the results of this paper. While such terms can change the location of the multicritical point [18, 19], the universal properties of the latter should not be affected by the details of the microscopic model.

We close this section by a discussion of the sign of various terms (e.g., the negative cross-Kerr nonlinearity) arising in the proposals of Refs. [18, 19, 101]. While a negative interaction term could lead to unbounded energy spectra, it would not pose a problem in the context of driven-dissipative systems where the steady state is not concerned with a minimum-energy ground state. Furthermore, one can change the sign of various terms in the Hamiltonian of a driven-dissipative system with a proper mapping [102]. For example, by sending $\Omega \rightarrow -\Omega$ and $a \rightarrow -a$ on one of the two sublattices, the sign of $J$ can be changed while leaving the remaining terms fixed. Similarly, one can also map $H \rightarrow -H$ by taking the complex conjugate of the master equation, which, together with the previous mapping, allows an appropriate choice for the sign of $J, V$. Finally, the overall phase of $\Omega$ is
unimportant while the parameter $\Delta$ can be easily directly tuned to a desired sign.

VI. CONCLUSION AND OUTLOOK

In this work, we have considered an experimentally relevant driven-dissipative system where two distinct order parameters emerge that characterize a liquid-gas type transition (associated with the average density) as well as an antiferromagnetic transition (associated with the difference in the sublattice density). The two phase transitions coalesce and form a multicritical point where both transitions occur at the same time. We have investigated the nontrivial interplay of two order parameters at the multicritical point. Using a field-theoretical approach—appropriate in the vicinity of the phase transition—we have shown that the critical behavior at this point can be mapped to a non-equilibrium stochastic model described by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Using perturbative renormalization group techniques, we have determined the RG flow equations of the model and identified a pair of new non-equilibrium fixed points which exhibit several exotic properties. First, we obtain two different exponents for the critical scaling of fluctuations and dissipation at the critical point, underscoring the violation of the fluctuation-dissipation relation at all scales and resulting in a behavior where the system becomes hotter and hotter at larger and larger scales. Furthermore, these NEFPs are distinguished by the emergence of discrete scale invariance and a complex Liouvillian gap even close to the critical point. Additionally, the phase diagram near these multicritical points displays spiraling phase boundaries. The latter properties could be particularly useful in identifying these NEFPs in experiments.

While generic driven-dissipative phase transitions tend to have effective equilibrium dynamics, we have shown that the interplay between several order parameters (in this case, two) could very well lead to exotic non-equilibrium behavior. This perspective opens a new avenue to investigate and experimentally realize non-equilibrium phases and phase transitions in the context of driven-dissipative systems without relying on the engineering of complicated non-local or non-Markovian dissipation.

Future experimental and numerical studies into the NEFPs discussed in this work are crucial to develop a more complete understanding of their properties. Characterizing the discrete scale invariance, either in the dynamics or the form of the phase boundary, defines a particularly important direction. Investigating the possible emergence and the critical behavior of such non-equilibrium phase transitions in low dimensions is worthwhile. It would be particularly interesting to identify low-dimensional ordering and phase transitions which are not otherwise possible in equilibrium settings. Another question that remains open is the fate of the subspace $g_{12} g_{21} = 0$, namely if it contains new NEFPs. Beyond these non-equilibrium generalizations of model A systems, one can further consider similar non-equilibrium versions of other equilibrium universality classes. While we focused on the particular case of an experimentally relevant model with only two scalar order parameters, our analysis indicates that a large class of new non-equilibrium multicritical points are yet to be discovered. A natural extension of our work is to identify possibly new NEFPs in $O(N_1) \times O(N_1)$ models involving vector-like order parameters [62–68]. While a driven-dissipative condensate of polaritons has been investigated theoretically in detail [27, 28], recent experimental studies into condensate supersolids [103–107] can provide excellent platforms for probing any emergent NEFPs. In addition to the $U(1)$ symmetry of the condensate, the two coupled optical cavities can provide either an additional $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (corresponding to a lattice supersolid) or an approximate $O(2)$ symmetry (corresponding to a continuous supersolid).

ACKNOWLEDGMENTS


Appendix A: Langevin equations near the multicritical points

In this appendix, we present the details of the derivation of the Langevin equations in the main text. To this end, we follow the procedure detailed in Ref. [15]. We begin by constructing the Keldysh path integral, then identify a semi-classical limit, and derive a pair of complex Langevin equations that describe the dynamics near the steady state. Finally, we identify a pair of two massless real fields (i.e., soft modes) and two massive real fields (i.e., fast modes). We adiabatically eliminate the massive fields to obtain a pair of Langevin equations presented in the main text.

We first ignore the sublattice symmetry for simplicity; this would not affect the analysis presented here. We shall return to the latter symmetry once we identify the semi-classical limit and corresponding Langevin equations. We cast our model in terms of a Keldysh path integral as

$$Z = \int \mathcal{D}[\psi_q, \psi_{\bar{q}}] e^{iS_K}, \quad (A1)$$
where the action \( S_K \) is defined as
\[
S_K = \int_{x,t} \psi_d^* \partial_t \psi_d + \psi_q \partial_t \psi_q^* \\
- \int_{x,t} (H_n(\psi_d + \psi_q) - H_n(\psi_d - \psi_q)) \\
+ i \Gamma \int_{x,t} (|\psi_d|^2 - |\psi_d|^2) - 2 - |\psi_d|^2 |\psi_d|^2 / 2),
\]
with \( \psi_d, \psi_q \) the classical/quantum fields and \( H_n(\psi) \) the normal-ordered form of the Hamiltonian. The third line corresponds to the particular case of the Lindbladian \( \sqrt{\Gamma} a \), although this approach may easily be extended to more general Lindbladians [30]. With the Hamiltonian in Eq. (1), the action in the continuum (with the nearest-neighbor interactions expanded in powers of the gradient) is given by
\[
S_K = \int_{x,t} \psi_q^* \left( i \partial_t + J \nabla^2 + \Delta + i \beta J + i \Gamma / 2 \right) \psi_d + c.c. \\
- \int_{x,t} V|\psi_d|^2 (\nabla^2 + \beta) \psi_d \psi_q^* - \sqrt{2} \Omega \psi_q^* + c.c. \\
+ \int_{x,t} i \Gamma |\psi_q|^2 - V|\psi_q|^2 (\nabla^2 + \beta) \psi_d \psi_q^* + c.c.,
\]
where \( \beta = 2d \) is the coordination number. This expression bears a close resemblance to the action of Eq. (12) in Ref. [15], but they differ in the form of their interactions (which involve gradient terms here) and due to our use of normal ordering rather than the Weyl ordering of Ref. [15]. This motivates a similar rescaling of the parameters as
\[
\psi_d = \psi_d / \sqrt{N}, \quad \psi_q = \psi_q / \sqrt{N}, \\
\Omega = \Omega / \sqrt{N}, \quad v = \beta V N.
\]
The parameter \( N \) effectively describes a density scale for the microscopic model via \( |\psi_d|^2 = N |\psi_d|^2 \). Since varying the density scale also modifies the interaction energy per particle, the interaction strength should be reduced correspondingly such that \( V|\psi_c|^2 = v|\psi_c|^2 \); similarly, the drive should be increased so that \( \Omega \psi_q = \Omega \psi_q \). We can then rewrite the action as
\[
S_K = \int_{x,t} \psi_q^* \left( i \partial_t + J \nabla^2 + \Delta + i \beta J + i \Gamma / 2 \right) \psi_d + c.c. \\
- \int_{x,t} v|\psi_d|^2 (\nabla^2 + \beta) \psi_d \psi_q^* - \sqrt{2} \Omega \psi_q^* + c.c. \\
+ \int_{x,t} i \beta |\psi_q|^2 - \frac{v}{N^2} |\psi_q|^2 (\nabla^2 + \beta) \psi_d \psi_q^* + c.c.
\]
In the limit of large \( N \), the last term (the second term in the last line) can be dropped, leading to an action that is at most quadratic in \( \psi_q \). This is simply because a large population \( N \) corresponds to the semi-classical limit represented by a large classical field \( \psi_d \) and small fluctuations due to the quantum field \( \psi_q \). Using this fact, we can map the action to a Langevin equation as [53]
\[
i \partial_t \Psi = - (J \nabla^2 + \Delta + 3 J + i \Gamma / 2 - v(1 + \nabla^2 / \beta)|\Psi|^2) \Psi + \Omega + \xi,
\]
where \( \xi \) is a random force.

Note that the noise level is further suppressed at larger \( N \) as should be expected from our semi-classical treatment.

Next we include the sublattice symmetry by defining \( \Psi_1 \) as the sublattice average and \( \Psi_2 \) as the sublattice difference of the field \( \Psi \); see Eq. (15), which differs by a factor of \( \sqrt{N} \) due to our semi-classical limit. Our new Langevin equations are now
\[
i \partial_t \Psi_1 = -(\Delta + J + i \Gamma / 2) \Psi_1 + v(\Psi_1^2 - \Psi_2^2) \Psi_1^* + \Omega + \xi_1,
\]
\[
i \partial_t \Psi_2 = -(\Delta - J + i \Gamma / 2) \Psi_2 + v(\Psi_2^2 - \Psi_1^2) \Psi_2^* + \xi_2,
\]
\[
\langle \xi_1^* (x,t) \xi_1(x',t') \rangle = \frac{\Gamma}{\mathcal{N}} \delta(x-x') \delta(t-t'), \quad \langle \xi_i \rangle = 0.
\]
Notice that we have dropped all the gradient terms as they do not play a role in identifying the massive fields and their adiabatic elimination. We also follow our convention in the main text to set \( 3 J \rightarrow J \).

Our model exhibits two multicritical points where two modes (each a component of one of the two fields \( \Psi \)) become critical. Due to the sublattice symmetry, \( \Psi_2 = 0 \) at the multicritical points up to fluctuations. Working in units where \( \Delta + J = 1 \), the two multicritical points occur at
\[
(\Delta_c, J_c) = \left( \frac{1}{3}, \frac{2}{3} \right), \quad \left( \frac{1}{3}, \frac{1}{3} \right),
\]
where \( \Psi_1 = \Psi_e = \sqrt{2 / 3} e^{-i \pi / 3} \).

Next, we expand the Langevin equations in the vicinity of the two multicritical points as
\[
\Delta = \Delta_c + \delta \Delta, \quad J = J_c - \delta \Delta, \\
\Gamma = \Gamma_c + \delta \Gamma, \quad \Omega = \Omega_c + \delta \Omega / \sqrt{v},
\]
\[
\Psi_1 = \Psi_1 + \psi_1 / \sqrt{v}, \quad \Psi_2 = \psi_2 / \sqrt{v}.
\]
The soft and gapped modes can be determined as linear combinations of the real and imaginary parts of the two fields \( \psi_1, \psi_2 \). As in the case of bistability, the first pair is identified as [15]
\[
\psi_1 = \phi_1 + e^{i \pi / 3} \phi_1,
\]
where \( \phi_1 \) is massive and relaxes quickly while \( \phi_1 \) defines the slow field. Identifying the massive/massless components of the field \( \psi_2 \) depends on the corresponding multicritical point as
\[
\Delta_c = 1 / 3: \quad \psi_2 = \frac{1}{\sqrt{3}} (\phi_1 e^{-i \pi / 6} + \phi_2 e^{i \pi / 6}),
\]
\[
\Delta_c = 2 / 3: \quad \psi_2 = \frac{1}{\sqrt{3}} (\phi_2 + \phi_2 e^{i \pi / 3}),
\]
where again the primed (unprimed) field indicates the massive (massless) field. Note that these differ from the main text by a factor of $\sqrt{V}$, which is done to simplify the resulting parameters in the Langevin equations by moving all the $V$ dependence to the noise term.

Upon adiabatically eliminating the massive fields and restoring the gradient terms, we arrive at the Langevin equations

$$
\dot{\phi}_1 = -h - r_1 \phi_1 + D_1 \nabla^2 \phi_1 + \xi_1 + A_{20} \phi_1^2 + A_{02} \phi_2^2 + A_{12} \phi_1 \phi_2 + A_{30} \phi_1^3, \tag{A12a}
$$

$$
\dot{\phi}_2 = -r_2 \phi_2 + D_3 \nabla^2 \phi_2 + \xi_2 + B_{11} \phi_1 \phi_2 + B_{21} \phi_1^2 \phi_2 + B_{30} \phi_2^3, \tag{A12b}
$$

with Gaussian noise

$$
\langle \xi_i(x, t) \xi_j(x', t') \rangle = \frac{2 \kappa_i}{N} \delta_{ij} \delta(x - x') \delta(t - t'). \tag{A13}
$$

The various numerical factors are summarized in Table II for the two multicritical points under consideration. Note that $\gamma_1 = 1$ and $T_1 = \kappa_i/N$.

## Appendix B: Redundant operators

In this appendix, we identify the redundant operators in the Langevin equations (A12). In general, this can be done at the level of the Schwinger-Keldysh action; however, we shall focus on the equivalent description in terms of the Langevin equations. This perspective is particularly suitable in dealing with the (Itô) regularization that is required to properly define the stochastic equations.

Consider a pair of Langevin equations that define an Itô process in the differential form [108]

$$
d\phi_1 = f_1(\phi_1, \phi_2)dt + \sqrt{\kappa_1}dW_1, \tag{B1a}
$$

$$
d\phi_2 = f_2(\phi_1, \phi_2)dt + \sqrt{\kappa_2}dW_2, \tag{B1b}
$$

where $dW_i$ is the stochastic noise that obeys the Itô rules:

$$
dW_idW_j = \delta_{ij}dt, \tag{B2a}
$$

$$
dW_idt = dt dW_i = 0, \tag{B2b}
$$

$$
dt^2 = 0. \tag{B2c}
$$

At the multicritical point, where the effective masses and the magnetic field are set to zero, the Langevin equations (A12) can be written in the form of Eq. (B1) with

$$
f_1(\phi_1, \phi_2) = D_1 \nabla^2 \phi_1 + A_{20} \phi_1^2 + A_{02} \phi_2^2 + A_{12} \phi_1 \phi_2 + A_{30} \phi_1^3, \tag{B3a}
$$

$$
f_2(\phi_1, \phi_2) = D_3 \nabla^2 \phi_2 + B_{11} \phi_1 \phi_2 + B_{21} \phi_1^2 \phi_2 + B_{30} \phi_2^3. \tag{B3b}
$$

In order to identify the redundant operators, we should examine the Langevin equations under a general change of the field variables. We should then find the dynamics in terms of new variables defined as $\Phi_1 = g_1(\phi_1, \phi_2)$ and $\Phi_2 = g_2(\phi_1, \phi_2)$; the functions $g_i$ are general (but local) nonlinear maps which are invertible in a neighborhood around the multicritical point and preserve the sublattice symmetry $\phi_2 \rightarrow -\phi_2$. The equations governing the dynamics of the new variables take the form

$$
d\Phi_1 = \frac{\partial g_1}{\partial \phi_1} d\phi_1 + \frac{\partial g_1}{\partial \phi_2} d\phi_2 + \frac{1}{2} \frac{\partial^2 g_1}{\partial \phi_1^2} d\phi_1^2 + \frac{1}{2} \frac{\partial^2 g_1}{\partial \phi_2^2} d\phi_2^2 + \frac{\partial g_1}{\partial \phi_1} \frac{\partial g_1}{\partial \phi_2} d\phi_1 d\phi_2
$$

$$
= \left( f_1 \frac{\partial g_1}{\partial \phi_1} + f_2 \frac{\partial g_1}{\partial \phi_2} + \frac{\kappa_1}{2} \frac{\partial^2 g_1}{\partial \phi_1^2} + \frac{\kappa_2}{2} \frac{\partial^2 g_1}{\partial \phi_2^2} \right) dt + \sqrt{\kappa_1} \frac{\partial g_1}{\partial \phi_1} dW_1 + \sqrt{\kappa_2} \frac{\partial g_1}{\partial \phi_2} dW_2. \tag{B4}
$$

We have used the Itô rules to derive the above equation, which is known as Itô’s formula or Itô’s lemma [108]. A similar stochastic equation can be derived for $\Phi_2$ by switching $1 \leftrightarrow 2$. Note that the terms on the RHS should be expressed in terms of $\Phi_1$ through the inverse functions $g_i^{-1}$. One notices that there are new contributions to the

<table>
<thead>
<tr>
<th>$\Delta_c = 1/3$</th>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
<th>$\frac{1}{3} D_1$</th>
<th>$\frac{1}{3} D_2$</th>
<th>$h$</th>
<th>$r_1$</th>
<th>$A_{20}$</th>
<th>$A_{02}$</th>
<th>$A_{12}$</th>
<th>$r_2$</th>
<th>$B_{11}$</th>
<th>$B_{03}$</th>
<th>$B_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_c = 2/3$</td>
<td>$\kappa_1$</td>
<td>$\kappa_2$</td>
<td>$\frac{1}{3} D_1$</td>
<td>$\frac{1}{3} D_2$</td>
<td>$h$</td>
<td>$r_1$</td>
<td>$A_{20}$</td>
<td>$A_{02}$</td>
<td>$A_{12}$</td>
<td>$r_2$</td>
<td>$B_{11}$</td>
<td>$B_{03}$</td>
<td>$B_{21}$</td>
</tr>
</tbody>
</table>
deterministic dynamics due to the noise. However, since we are working under the assumption that the noise is small (with a strength proportional to $1/N$), we can ignore such terms. Additionally, the noise terms are no longer additive but are instead multiplicative, which introduces new terms in the action. However, the latter are irrelevant in the sense of RG and can be neglected as well. The same holds for nonlinear terms (beyond quadratic terms) that involve gradients.

We shall assume without loss of generality that 

$$\frac{\partial g_i}{\partial \phi_j} \big|_{\phi = 0} = \delta_{ij};$$  \hfill (B5)

rescaling the fields by a constant factor does not allow us any additional freedom while rotations obscure the symmetry $\phi_2 \to -\phi_2$. Additionally, we do not consider a constant shift in the field $\phi_1$ (a shift in $\phi_2$ is disallowed due to symmetry) for now, but discuss it separately later in this appendix. Based on the structure of Eq. (B4), we notice that the quadratic terms in $f_i$ and $g_i$ result in additional cubic terms. All other new terms in the deterministic part of the dynamics involve fourth- or higher-order terms which are irrelevant under RG. Expressing a general nonlinear transformation as 

$$g_1(\phi_1, \phi_2) = \phi_1 + c_{20} \phi_1^2 + c_{02} \phi_2^2,$$ \hfill (B6a)

$$g_2(\phi_1, \phi_2) = \phi_2 + c_{11} \phi_1 \phi_2,$$ \hfill (B6b)

the modification of the cubic terms to lowest order in the coefficients and interaction terms are given by

$$
\begin{pmatrix}
A_{12} \\
B_{21} \\
B_{03}
\end{pmatrix} =
\begin{pmatrix}
A_{12} \\
B_{21} \\
B_{03}
\end{pmatrix} + M
\begin{pmatrix}
c_{20} \\
c_{02} \\
c_{11}
\end{pmatrix},$$

\hfill (B7)

where the matrix $M$ is given by

$$M = \begin{pmatrix}
2A_{02} & 2B_{11} - 2A_{20} & -2A_{02} \\
-B_{11} & 0 & A_{20} \\
0 & -B_{11} & A_{02}
\end{pmatrix}. \hfill (B8)
$$

Notice that the coefficient $A_{30}$ is left unchanged. The rank of matrix $M$ is 2, which then determines the number of corresponding redundant operators.

In addition to the two redundant operators above, a third one emerges due to a constant shift $\phi_1 \to \phi_1 + c_{00}$. Under this transformation, the quadratic terms transform as

$$
\begin{pmatrix}
A_{20} \\
A_{02} \\
B_{11}
\end{pmatrix} =
\begin{pmatrix}
A_{20} \\
A_{02} \\
B_{11}
\end{pmatrix} + \begin{pmatrix}
3A_{30} \\
A_{12} \\
4A_{21}
\end{pmatrix} c_{00}. \hfill (B9)
$$

The effective mass and magnetic field terms also change, but this simply shifts the location of the critical point.

The three redundant operators derived here can be used to always set the terms $A_{20}, A_{02}, B_{11}$ to zero. This can be understood by noting that the transformation corresponding to $M$ allows one to adjust the ratios of $A_{12}$ and $A_{31}$ relative to $A_{30}$ without changing the quadratic terms. By properly using this redundancy, these ratios can be tuned until the constant shift in Eq. (B9) shifts all three quadratic terms to zero. At the same time, the cubic terms transform as

$$A_{30} \to A_{30}, \hfill (B10a)$$

$$B_{03} \to B_{03} - \frac{A_{12} A_{20} B_{11} - 6 A_{02} A_{30} B_{11} + 2 A_{02} A_{20} B_{21}}{2 A_{20} (A_{20} - B_{11})}, \hfill (B10b)$$

$$A_{12} \to 2 A_{02} \frac{3 A_{30}}{2 A_{20}}, \hfill (B10c)$$

$$B_{21} \to B_{11} \frac{3 A_{30}}{2 A_{20}}. \hfill (B10d)$$

Having exhausted the three redundant operators to remove the three quadratic terms, there is no further freedom in tuning other terms and, specifically, all the cubic terms are fixed. While we could in principle include cubic or higher-order terms in the nonlinear transformation [Eq. (B6)], these would only modify the fourth- or higher-order terms which are irrelevant under RG. Finally, we note that the coefficient $A_{30}$ appears in the denominator of the above transformations. We assumed that this term is generated under coarse-graining and thus should pose no problem in making the above transformations. However, if there is a mechanism where this coefficient could be tuned to zero, the above transformations are no longer valid and the two nonzero cubic terms should be kept.

### Appendix C: Perturbative RG

In this appendix, we discuss the details of the calculations in our perturbative RG analysis. In the first part, we introduce the diagrammatic techniques we have used in the main text. In the second part, we compute the one-loop diagrams, while, in the third part, we compute the two-loop diagrams for terms which are unrenormalized at one loop.

#### 1. Diagrammatic techniques

To define the Gaussian propagators, we start with the Gaussian model with the corresponding action

$$A_0[\phi_i, \phi_i] = \sum_i \int_{t,x} \hat{\phi}_i(\gamma_i \hat{\phi}_i - D_i \nabla^2 + r_i)\phi_i - \gamma_i T_i \hat{\phi}_i^2.$$ \hfill (C1)
the one-loop contribution to the renormalization of these divergences in the evaluation of integrals in the fol-
ations, which are illustrated in Fig. 10. The combinatorial factors are (a) $3\phi_1\phi_1$, (b) $u_2\phi_2\phi_2$, (c) $u_12\phi_1\phi_1\phi_1$, (d) $\sigma u_12\phi_1\phi_2\phi_2$.

![Figure 8](image1)

**FIG. 8.** Diagrammatic representation of Gaussian propagators. Solid (dotted) lines correspond to classical (response) fields. The two fields are later distinguished by thickness and color. In this figure, we have shown (a) the response propagator, and (b) the correlation propagator.

![Figure 9](image2)

**FIG. 9.** Interaction vertices. Thin black (thick cyan) lines correspond to the first (second) field and solid (dashed) lines correspond to the classical (response) field. (a) $u_1\phi_1\phi_1$. (b) $u_2\phi_2\phi_2$. (c) $u_12\phi_1\phi_1\phi_1$. (d) $\sigma u_12\phi_1\phi_2\phi_2$.

The Gaussian response and correlation functions are then given by

$$\chi_{0}^i(q, \omega) = \mathcal{F}(\phi_i(0,0)\phi_i(r, t)) = \frac{1}{-i\gamma + D_i q^2 + r_i}, \quad (C2a)$$

$$C_{0}^i(q, \omega) = \mathcal{F}(\phi_i(0,0)\phi_i(r, t)) = \frac{2\gamma T_i}{\gamma^2 \omega^2 + (D_i q^2 + r_i)^2}, \quad (C2b)$$

where $\mathcal{F}$ denotes the Fourier transform in both space and time. These propagators can be expressed in a diagrammatic representation as shown in Fig. 8.

The four interaction vertices from Eq. (27b) are illustrated in Fig. 9. Due to the structure of the action, one can find the corresponding $Z$ factors for $\phi_2$ by switching the subscripts $1 \leftrightarrow 2$ and multiplying $u_{12}$ by a factor of $\sigma$.

Finally, we quote identities which will prove useful in computing the integrals in various Feynman diagrams:

$$\frac{1}{AV^B} = \frac{\Gamma(r + s)}{\Gamma(r)\Gamma(s)} \int_0^1 \frac{x^{r-1}(1-x)^{s-1}}{x^A + (1-x)^B} dx, \quad (C3a)$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(m^2 + 2q \cdot p + Dp^2)^s} = \frac{\Gamma(s-d/2)}{(4\pi)^{d/2}\Gamma(s)(m^2 - q^2/D)^{s-d/2}}, \quad (C3b)$$

where $\Gamma(x)$ is Euler’s Gamma function. Finally, in order to determine the $Z$ factors, we employ the minimal subtraction procedure. This means that only the ultraviolet divergences, in the form of powers of $1/\epsilon$, are incorporated into the $Z$ factors. For simplicity, we only present these divergences in the evaluation of integrals in the following sections; non-divergent terms are dropped.

### 2. One-loop diagrams

#### a. Mass terms

In this section, we consider corrections to $r_1$. The corrections to $r_2$ can be obtained by switching the roles of $\phi_1$ and $\phi_2$. There are two diagrams that provide corrections, which are illustrated in Fig. 10. The combinatorial and interaction factors are (a) $3 \times u_1$ and (b) $u_{12}$. Thus the one-loop contribution to the renormalization of $r_1$ is

$$\int \frac{d^d p}{(2\pi)^d} \int \frac{d\omega}{2\pi} \left[ \beta u_1 C_0^1(p, \omega) + u_{12} C_0^2(p, \omega) \right], \quad (C4)$$

so we should evaluate the integral of $C_0^i$. The latter can be written as

$$2\gamma T_i \int \frac{d^d p}{(2\pi)^d} \int \frac{d\omega}{2\pi} \frac{1}{\gamma_1^2 \omega^2 + (D_1 p^2 + r_i)^2}$$

$$= T_i \int \frac{d^d p}{(2\pi)^d} \frac{1}{D_1 p^2 + r_i}, \quad (C5)$$

where the last line follows once we integrate over frequency.

The phase transition occurs where the renormalized mass term vanishes. The critical value of the mass parameter $r_1$ is then determined by

$$r_1 = -3u_1 T_i \int \frac{d^d p}{(2\pi)^d} \frac{1}{D_1 p^2} - u_{12} T_2 \int \frac{d^d p}{(2\pi)^d} \frac{1}{D_2 p^2}, \quad (C6)$$
and similarly for $r_2$; note that the factors of $r$ in the denominator have been dropped as they introduce $O(u^2)$ corrections. Defining an additive renormalized mass term $\tau_i = r_i - \bar{r}_i$, we can determine the $Z$ factors for $\tau_i$. To this end, we need to compute integrals of the form

$$ \int \frac{d^d p}{(2\pi)^d} \frac{\tau_i}{D_i^2} = \frac{A_{d\mu}}{\epsilon} \frac{\tau_i}{D_i^2}, \quad (C7) $$

where we have included the geometrical factor $A_d$. We can then compute the corresponding $Z$ factors as

$$ Z_{\tau_1} = 1 - 3 u_1 - \frac{A_{d\mu}}{\epsilon} \frac{T_1}{D_1^2} - u_{12} \frac{A_{d\mu}}{\epsilon} \frac{T_2}{D_2^2} \tau_2, \quad (C8a) $$

$$ Z_{\tau_2} = 1 - 3 u_2 - \frac{A_{d\mu}}{\epsilon} \frac{T_2}{D_2^2} - u_{12} \frac{A_{d\mu}}{\epsilon} \frac{T_1}{D_1^2} \tau_1. \quad (C8b) $$

From this point on, we simply write $\tau_i$ as $r_i$.

### b. Coupling terms

We first consider one-loop corrections to $u_1$. We need to consider two diagrams as illustrated in Fig. 11. The combinatorial and interaction factors are (a) $-3 \times 2 \times 3 \times u_1^2$, (b) $-2 \times 2 \times \sigma u_{12}^2$, (c) $-2 \times 3 \times u_2 \times \sigma u_{12}$, and (d) $-3 \times 2 \times u_1 \times u_{12}$. The remaining four diagrams, corresponding to the renormalization of $\sigma u_{12}^2 \phi_2^2 \phi_2^2$, can be simply obtained by interchanging the two fields. The resulting set of internal diagrams and combinatorial factors are the same up to a factor of $\sigma$, so $\phi_2^2 \phi_1 \phi_1$ and $\phi_2^2 \phi_2^2$ are renormalized in the same way at this order. Indeed, this reflects the fact that $g_{12}/g_{21}$ is not renormalized at the one-loop order.

Thus the one-loop contribution to the renormalization of $u_{12}$ is

$$ - \int \frac{dp}{(2\pi)^d} \int \frac{d\omega}{2\pi} \left[ 18 u_{12}^2 \chi_0^1(p, \omega) C_0^1(-p, -\omega) + 2 \sigma u_{12}^2 \chi_0^2(p, \omega) C_0^2(-p, -\omega) \right]. \quad (C9) $$

This expression involves a nontrivial integral of the form

$$ U_{ij} = - \int \frac{dp}{(2\pi)^d} \int \frac{d\omega}{2\pi} \frac{1}{\gamma_i \gamma_j} \frac{1}{2 \gamma_i T_j} \frac{1}{(D_i p^2 + r_i \gamma_i \omega + D_j p^2 + r_j \gamma_j \omega + (D_i + D_j) p^2 + x \gamma_i \gamma_j + \gamma_i \gamma_j + x \gamma_i \gamma_j)^2}. \quad (C10) $$

Integrating out frequency, the latter integral becomes

$$ - \frac{T_i}{\gamma_i \gamma_j} \int \frac{dp}{(2\pi)^d} \frac{1}{(D_i p^2 + \tilde{r}_i)((D_i + D_j) p^2 + \gamma_j \tilde{r}_i + \gamma_i \tilde{r}_j)}, \quad (C13) $$

where $D_i = \gamma_i \tilde{D}_i$ and $r_i = \gamma_i \tilde{r}_i$. Using Feynman’s parametrization, this may be rewritten

$$ - \frac{T_j}{\gamma_i \gamma_j} \int \frac{dp}{(2\pi)^d} \frac{1}{[(1 - x)(D_j p^2 + \tilde{r}_j) + x(D_i + D_j) p^2 + x \tilde{r}_i + x \tilde{r}_j]^2}, \quad (C14) $$

allowing us to integrate over momentum to find

$$ - \frac{T_i}{\gamma_i} \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}} \int_0^1 dx \frac{1}{(\tilde{r}_i + x \tilde{r}_j)^{d/2}}. \quad (C15) $$

Noting that $\tilde{r}_i = \mu^2 \tilde{r}_{1i} + O(u)$ and $\tilde{r}_{1i}$ is a finite constant, then according to the minimal subtraction pro-
Integrate over momenta, first combining the second two factors in the \( \epsilon \) procedure, (\( \tilde{r}_j x \)) and the integral becomes 

\[
\sigma \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \chi_0^2 (k - p - q, \omega - \omega_1 - \omega_2) C_0(p, \omega_1) C_0^*(q, \omega_2),
\]

result in combination with the diagrams considered above results in the following \( Z \) factors

\[
Z_{u_1} = 1 + 18U_{11} u_1 + 2U_{22} u_1^2 + u_2, \quad \text{(C17a)}
\]

\[
Z_{u_2} = 1 + 18U_{22} u_2 + 2U_{11} u_2^2 + u_1, \quad \text{(C17b)}
\]

\[
Z_{u_{12}} = 1 + 4U_{21} u_{12} + 4U_{12} u_{12} + 6U_{11} u_1 + 6U_{22} u_2, \quad \text{(C17c)}
\]

3. Two-loop diagrams

First, we consider the two-loop corrections which arise from the \( \phi_i \phi_i \) terms. There are three two-loop diagrams that renormalize \( \gamma_1 \) and \( D_1 \) as shown in Fig. 13. The corresponding combinatorial and interaction factors are (a) \(-\frac{\gamma}{2\pi} \times 3 \times 3 \times u_1^2\), (b) \(-\frac{\gamma}{2\pi} \times 2 \times u_1^2\), and (c) \(-2 \times 2 \times u_2^2\).

As in the case of the coupling terms, the internal diagrams are all of the same form, so we consider a generic internal diagram in the form of Fig. 14.

This diagram corresponds to the integral

\[
I_{ij}^k = \int \frac{d^d p d^d q}{(2\pi)^{2d}} \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \chi_0^2 (k - p - q, \omega - \omega_1 - \omega_2) C_0(p, \omega_1) C_0^*(q, \omega_2),
\]

or

\[
\int \frac{d^d p d^d q}{(2\pi)^{2d}} \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \frac{4\gamma \gamma_1 \gamma_2 T_{ij} T_{ij}}{(D_i p^2 + r_i)(D_j q^2 + r_j)(D_i q^2 + D_j q^2 + D_k (k - p - q)^2 + r_i + r_j + r_k - i\omega)\}
\]

Integrating out the frequencies, this becomes

\[
\frac{T_{ij} T_{ij}}{\gamma_1 \gamma_2 \gamma_k} \int \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{2}{(D_i p^2 + r_i)(D_j q^2 + r_j)(D_i q^2 + D_j q^2 + D_k (k - p - q)^2 + r_i + r_j + r_k - i\omega)}.
\]

Once more, we use Feynman parameters to integrate over momenta, first combining the second two factors in the denominator as

\[
\frac{T_{ij} T_{ij}}{\gamma_1 \gamma_2 \gamma_k} \int \frac{d^d p}{(2\pi)^d} \frac{1}{D_i p^2 + r_i} \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{1}{(\alpha_0 + 2\alpha_1 \cdot q + \alpha_2 q^2)^2},
\]

where

\[
\alpha_0 = \tilde{r}_j (1 - x) + x (\tilde{D}_j q^2 + \tilde{D}_k (k - q)^2 + \tilde{r}_i + \tilde{r}_j + \tilde{r}_k - i\omega),
\]

\[
\alpha_1 = x \tilde{D}_k (p - k),
\]

\[
\alpha_2 = \tilde{D}_i + x \tilde{D}_k.
\]

Integrate over \( q \) yields

\[
\frac{T_{ij} T_{ij}}{\gamma_1 \gamma_2 \gamma_k} \Gamma(2 - d/2) \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \frac{1}{(D_i p^2 + r_i) (\alpha_0 - \alpha_1^2/\alpha_2)^{d/2}}.
\]
Again taking advantage of a second Feynman parameter, we write this as

$$I_{ij} = \frac{T_i T_j}{\gamma_i \gamma_j \gamma_k} \frac{\Gamma(2 - d/2) \Gamma(3 - d/2)}{(4\pi)^{d/2}} \int_0^1 \int_0^1 dxdy \int \frac{d^d p}{(2\pi)^d} \frac{\alpha_2^{-d/2} y^{1 - d/2}}{(\beta_0 + 2\beta_1 + \beta_2 p)^{3 - d/2}}, \quad (C24)$$

where

$$\beta_0 = (1 - y)\tilde{r}_i + y(\tilde{r}_j + x\tilde{D}_k)k^2 + \tilde{r}_k x + \tilde{r}_i x - i x \omega - \frac{x^2 \tilde{D}_k^2 k^2}{\tilde{D}_j + x\tilde{D}_k}, \quad (C25a)$$

$$\beta_1 = \frac{-x y \tilde{D}_k \tilde{D}_j}{\tilde{D}_j + x\tilde{D}_k}, \quad (C25b)$$

$$\beta_2 = (1 - y)\tilde{D}_i + x y \left( \tilde{D}_i + \frac{\tilde{D}_k \tilde{D}_j}{\tilde{D}_j + x\tilde{D}_k} \right). \quad (C25c)$$

Integrating over $p$, we are left with the expression

$$I_{ij} = \frac{T_i T_j}{\gamma_i \gamma_j \gamma_k} \frac{\Gamma(2 - d/2) \Gamma(3 - d/2)}{(4\pi)^{d/2}} \int_0^1 \int_0^1 dxdy \frac{(\alpha_2 \beta_2)^{-d/2} y^{1 - d/2}}{(\beta_0 - \beta_1^2/2)^{3 - d/2}}. \quad (C26)$$

In order to determine corrections to $\omega$ and $k^2$, we consider $W_{ij}^k = \frac{\partial T_{ij}}{\partial (\omega_k)}$ and $K_{ij}^k = \frac{\partial T_{ij}}{\partial (k^2)}$, respectively, in the limit $\omega \to 0, k \to 0$. Additionally, noting $\Gamma(3 - d) = -\frac{1}{x} + O(1)$ and taking $d \to 4$ while extracting a factor $\mu^{-2x}$, we have

$$W_{ij}^k = -\frac{T_i T_j}{4\gamma_i \gamma_j \gamma_k} \frac{A^2_{ij} \mu^{-2x}}{\epsilon} \int_0^1 \int_0^1 \frac{dx dy}{(\tilde{D}_k \tilde{D}_j x y + \tilde{D}_i (\tilde{D}_j + \tilde{D}_k x)(1 - y + x y))^2}, \quad (C27a)$$

$$K_{ij}^k = -\frac{T_i T_j}{4\gamma_i \gamma_j \gamma_k} \frac{A^2_{ij} \mu^{-2x}}{\epsilon} \int_0^1 \int_0^1 \frac{dx dy}{(\tilde{D}_k \tilde{D}_j x y + \tilde{D}_i (\tilde{D}_j + \tilde{D}_k x)(1 - y + x y))^3}, \quad (C27b)$$

where we have included the geometrical factor $A^2_{ij}$. We evaluate both of these integrals exactly to find the resulting corrections

$$W_{ij}^k = -\frac{T_i T_j}{4\gamma_i \gamma_j \gamma_k} \frac{A^2_{ij} \mu^{-2x}}{\epsilon} \frac{1}{D_k^2 D_i D_j} \log \left( \frac{(\tilde{D}_i + \tilde{D}_j)(\tilde{D}_k + \tilde{D}_j)}{D_i D_j + D_k(D_i + D_j)} \right), \quad (C28a)$$

$$K_{ij}^k = -\frac{T_i T_j}{4\gamma_i \gamma_j \gamma_k} \frac{A^2_{ij} \mu^{-2x}}{\epsilon} \frac{(\tilde{D}_i + \tilde{D}_j)\tilde{D}_k^2 + 2\tilde{D}_i \tilde{D}_j \tilde{D}_k}{2D_i D_j (D_i + D_k)(D_j + D_k)(D_i D_j + D_i D_k + D_j D_k)}, \quad (C28b)$$

from which we identify the following $Z$ factors

$$Z_{\gamma_2} = 1 - (18u_2^2 W_{11}^2 + 2u_2^2 W_{22}^2 + 4\sigma u_1^2 W_{12}^2)/\gamma_2, \quad (C29a)$$

$$Z_{\gamma_1} = 1 - (18u_2^2 W_{11}^2 + 2u_2^2 W_{22}^2 + 4\sigma u_1^2 W_{12}^2)/\gamma_1, \quad (C29a)$$
\[ Z_{D_1} = 1 - (18u_1^2K_{11}^2 + 2u_2^2K_{12}^2 + 4\sigma u_3^2K_{12}^2) / D_1, \quad (C29c) \]

\[ Z_{D_2} = 1 - (18u_2^2K_{22}^2 + 2u_1^2K_{11}^2 + 4\sigma u_3^2K_{21}^2) / D_2, \quad (C29d) \]

where the three corrections correspond to diagrams (a), (b), and (c) in Fig. 13, respectively.

Next, we consider the two-loop corrections to the \( \hat{\phi}_i^2 \) terms. There are two such diagrams, which are illustrated in Fig. 15. The combinatorial and interactions factors are (a) \( \frac{\alpha^2}{3!} \times u_2^2 \) and (b) \( \frac{\alpha^2}{3!} \times 2 \times u_1^2 \). Note the lack of minus sign due to the sign difference in \( A[\hat{\phi}_1, \phi_i] \).

Again, the internal diagrams are all of the same form, so we instead consider the generic internal diagram in Fig. 16. The integral corresponding to this diagram is

\[ S_{ijk} = \int \frac{d^dp d^dq}{(2\pi)^{2d}} \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \chi_0(p, \omega_1) \chi_0(q, \omega_2) \chi_0(p + q, \omega_1 + \omega_2), \quad (C30) \]

or

\[ \frac{T_i T_j T_k}{\gamma_i \gamma_j \gamma_k} \int \frac{d^dp d^dq}{(2\pi)^{2d}} \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \left[ (D_i p^2 + \hat{r}_i)^2 + \omega_1^2 \right] \left[ (D_j q^2 + \hat{r}_j)^2 + \omega_1^2 \right] \left[ (D_k (p + q)^2 + \hat{r}_k)^2 + (\omega_1 + \omega_2)^2 \right]. \quad (C31) \]

Integrating out the frequencies, this becomes

\[ \frac{T_i T_j T_k}{\gamma_i \gamma_j \gamma_k} \int \frac{d^dp d^dq}{(2\pi)^{2d}} \left( D_i p^2 + D_j q^2 + D_k (p + q)^2 + \hat{r}_i + \hat{r}_j + \hat{r}_k \right) \left( D_i p^2 + \hat{r}_i \right) \left( D_j q^2 + \hat{r}_j \right) \left( D_k (p + q)^2 + \hat{r}_k \right). \quad (C32) \]

As before, we use Feynman parametization to integrate over \( q \). However, since there are three terms in the denominator that involve \( q \), we should introduce two Feynman parameters \( x, y \). The integral then becomes

\[ 2 \frac{[T_i T_j T_k] \Gamma(3)}{\gamma_i \gamma_j \gamma_k (4\pi)^{d/2}} \frac{1}{\Gamma(3)} \int_0^1 \int_0^1 dxdy \int \frac{d^dp}{(2\pi)^d} \frac{d^dq}{(2\pi)^d} \frac{y}{(\alpha_0 + 2\alpha_1 \cdot q + \alpha_2 q^2)^3}, \quad (C33) \]

where

\[ \alpha_0 = (1 - y)(\hat{D}_i + \hat{D}_k)p^2 + xy\hat{D}_k p^2 + (1 - y)\hat{r}_i + (1 - xy)\hat{r}_j + (1 - y + xy)\hat{r}_k, \quad (C34a) \]

\[ \alpha_1 = \hat{D}_k(1 - y + xy)p, \quad (C34b) \]

\[ \alpha_2 = (1 - xy)\hat{D}_j + (1 - y + xy)\hat{D}_k. \quad (C34c) \]

Integrating over \( q \), we obtain

\[ 2 \frac{[T_i T_j T_k] \Gamma(3)}{\gamma_i \gamma_j \gamma_k (4\pi)^{d/2}} \frac{1}{\Gamma(3)} \int_0^1 \int_0^1 dxdy \int \frac{d^dp}{(2\pi)^d} \frac{d^dq}{(2\pi)^d} \frac{y^{\alpha_2 - d/2}}{(\alpha_0 - \alpha_1^2/\alpha_2)^{3-d/2}}. \quad (C35) \]

Introducing a third Feynman parameter, we write this as

\[ 2 \frac{[T_i T_j T_k] \Gamma(3)}{\gamma_i \gamma_j \gamma_k (4\pi)^{d/2}} \frac{1}{\Gamma(3)} \int_0^1 \int_0^1 dxdydz \int \frac{d^dp}{(2\pi)^d} \frac{d^dq}{(2\pi)^d} \frac{y^{z^2 - d/2} z^{-d/2}}{(\beta_0 + 2\beta_1 \cdot q + \beta_2 q^2)^{4-d/2}}, \quad (C36) \]

where

\[ \beta_0 = (1 - yz)\hat{r}_i + (1 - xy)z\hat{r}_j + (1 - y + xy)z\hat{r}_k, \quad (C37a) \]

\[ \beta_1 = 0, \quad (C37b) \]

\[ \beta_2 = (1 - yz)\hat{D}_i + (1 - y + xy)z\hat{D}_k - \frac{\hat{D}_k (1 - y + xy)^2 z}{\hat{D}_k (1 - y + xy) + \hat{D}_j (1 - xy)}. \quad (C37c) \]
the same, this integral can be evaluated analytically as a factor

Integrating over $q$ leaves us with the integral

$$2T_iT_jT_k \Gamma(3) \Gamma(3 - d/2) \Gamma(4 - d/2) \Gamma(4 - d) \int_0^1 \int_0^1 \int_0^1 dx dy dz \frac{y}{D_i D_k (-1 + y - xy^2 + x^2 y^2) z + D_i (1 - xy) + D_k (1 - y + xy) (-1 + y z)^2},$$

(C38)

Several of the $\Gamma$ factors cancel out; also, we note that $\Gamma(4 - d) = 1/\epsilon + O(1)$. Taking the limit $d \to 4$ and extracting a factor $\mu^{-2\epsilon}$, we rewrite the latter integral as

$$C = \frac{2T_i T_j T_k A^2 \mu^{-2\epsilon}}{2 \gamma_i \gamma_j \gamma_k},$$

(C39b)

where we have included the geometrical factor $A^2$. Taking advantage of the fact that at least two of the $\tilde{D}$ must be the same, this integral can be evaluated analytically as

$$S_{ij} = S_{ji} = \frac{T_i T_j T_k}{2 \gamma_i \gamma_j^2} \frac{A^2 \mu^{-2\epsilon}}{\epsilon} \frac{1}{D_i^2 D_j^2} \log \left( \frac{2D_i D_j}{D_i + D_j} \left( \frac{D_i + D_j}{2D_i + D_j} \right)^{1 + 2D_i / D_j} \right),$$

(C40)

where $D_i$ corresponds to the field with one propagator and $\tilde{D}_j$ to the field with two.

Thus we identify the corresponding $Z$ factors

$$Z_{\gamma_1 T_1} = 1 + (3u_1^2 S_1 + u_2^2 S_1^2) / (\gamma_1 T_1),$$

(C41a)

$$Z_{\gamma_2 T_2} = 1 + (3u_2^2 S_2^2 + u_2^2 S_1^2) / (\gamma_2 T_2),$$

(C41b)

where the two corrections correspond to diagrams (a) and (b) in Fig. 15, respectively. Note that the factors are half of their combinatorial factors. This is because the zeroth order vertex is $2\gamma_i T_i$ rather than $\gamma_i T_i$.

**Appendix D: Method of characteristics**

In this section, we employ the method of characteristics in order to derive the scaling behavior of the correlation and response functions at or near a given fixed point. Since the correlation and response functions do not depend on the renormalized parameters, they are independent of the momentum scale of renormalization $\mu$. Thus,

$$\mu \frac{d}{d\mu} C_i(\mu, \omega, \mu^2, \{p_R\}, \{u_R\}) = 0,$$

(D1a)

$$\mu \frac{d}{d\mu} \chi_i(\mu, \omega, \mu^2, \{p_R\}, \{u_R\}) = 0,$$

(D1b)

where $\{p\} = \{\gamma_i, D_i, T_i, r_i\}$ and $\{u\} = \{u_1, u_2, u_{12}\}$ are the interaction strengths. Additionally, we define dimensionless scaling functions via $C_i = \mu^{-2} \tilde{C}_i$ and $\chi_i = \mu^{-2} \tilde{\chi}_i$, where the scaling factors are due to the scaling dimensions of the fields as well as the delta functions—which are factored out—corresponding to momentum and energy conservation. In each case, we can rewrite the total derivative $\mu \frac{d}{d\mu}$ in terms of the partial deriva-
Next we employ the method of characteristics and define parameters \( \hat{t} \) with respect to other parameters as

\[
\mu \frac{d}{d\mu} C_i = \mu^{-4} \left( \sum_p \gamma_p p R \partial_{p R} + \sum_s \beta_u \partial_{u R} - 4 \right) \hat{C}_i, \tag{D2a}
\]

\[
\mu \frac{d}{d\mu} \chi_i = \mu^{-2} \left( \sum_p \gamma_p p R \partial_{p R} + \sum_s \beta_u \partial_{u R} - 2 \right) \hat{\chi}_i, \tag{D2b}
\]

Casting the correlation and response functions in terms of the new flowing parameters and \( l \), the scaling functions satisfy the differential equations

\[
\left( \frac{d}{dl} - 4 \right) \hat{C}_i(\mathbf{q}/(\mu l), \omega/(\mu l)^2, \{ \hat{p}(l) \}, \{ \hat{u}(l) \}) = 0, \tag{D5a}
\]

\[
\left( \frac{d}{dl} - 2 \right) \hat{\chi}_i(\mathbf{q}/(\mu l), \omega/(\mu l)^2, \{ \hat{p}(l) \}, \{ \hat{u}(l) \}) = 0. \tag{D5b}
\]

At the fixed point, \( \beta_u(l) = 0 \) for all \( u \) and the parameters and flow functions assume their fixed-point values \( \hat{u}(l) = u^* \) and \( \gamma_p(l) = \gamma_p^* \). This allows us to easily solve the flowing parameters as

\[
\hat{p}(l) = p R l^{\gamma_p^*}. \tag{D6}
\]

Thus, at the fixed point, we can solve Eq. (D5) and find the scaling form of the correlation and response functions as

\[
C_i(\mathbf{q}, \omega, \{ p R \}) = (\mu l)^{-4} \hat{C}_i(\mathbf{q}/(\mu l), \omega/(\mu l)^2, \{ p R l^{\gamma_p^*} \}), \tag{D7a}
\]

\[
\chi_i(\mathbf{q}, \omega, \{ p R \}) = (\mu l)^{-2} \hat{\chi}_i(\mathbf{q}/(\mu l), \omega/(\mu l)^2, \{ p R l^{\gamma_p^*} \}). \tag{D7b}
\]

where the \( u \) arguments have been dropped since they approach a constant at the fixed point and do not affect the universal scaling behavior. Since all the terms in the perturbation series involve integrals of Gaussian propagators, we can simplify reduce the scaling functions to [cf. Eq. (C2)],

\[
\hat{C}_i \left( \frac{\mathbf{q}}{\mu l}, \frac{\omega}{(\mu l)^2}, D R l^{\gamma_p}, T R l^{\gamma_T}, \gamma_{i n}^{\gamma}, \{ r_{j n}^{\gamma} \} \right) \rightarrow \frac{\gamma_{i n} T R}{D R^2} l^{\gamma_p + \gamma_T - 2\gamma_B} \hat{C}_i \left( \frac{\mathbf{q}}{\mu l}, \frac{\gamma_{i n} \omega}{\mu^2 D R^2} l^{-2-\gamma_B + \gamma_T}, 1, 1, \{ r_{j n}^{\gamma} \gamma_T - \gamma_B \} \right), \tag{D8a}
\]

\[
\hat{\chi}_i \left( \frac{\mathbf{q}}{\mu l}, \frac{\omega}{(\mu l)^2}, D R l^{\gamma_p}, T R l^{\gamma_T}, \gamma_{i n}^{\gamma}, \{ r_{j n}^{\gamma} \} \right) \rightarrow D R^{-1} l^{-\gamma_B} \hat{\chi}_i \left( \frac{\mathbf{q}}{\mu l}, \frac{\gamma_{i n} \omega}{\mu^2 D R^2} l^{-2-\gamma_B + \gamma_T}, 1, 1, \{ r_{j n}^{\gamma} \gamma_T - \gamma_B \} \right), \tag{D8b}
\]

where we have utilized the fact that the scaling behavior of \( \gamma, D, T \) is the same for both fields. The above simplification was made by noting that in the Gaussian propagators, we can absorb some arguments into others, e.g.,

\[
C_i(\mathbf{q}, \omega, \{ r_{j} \}) \propto |\mathbf{q}|^{-4+\gamma_T+\gamma_T-2\gamma_B} \hat{C}_i \left( \frac{\mathbf{q}}{|\mathbf{q}|^{\gamma_T+\gamma_T-\gamma_B}}, \left\{ \frac{r_{j n}^{\gamma}}{|\mathbf{q}|^{\gamma_T+\gamma_T-\gamma_B}} \right\} \right), \tag{D9a}
\]
\[ \chi_i(q, \omega, \{r_j\}) \propto |q|^{-2-\gamma_D^*} \chi_i \left( \frac{\omega}{|q|^{2+\gamma_D^*}} \left\{ \frac{r_{jR}}{|q|^\gamma_j + \gamma_D^*} \right\} \right) , \]  

(D9b)

where we have further simplified the arguments of the scaling functions by dropping factors of \( \mu \) and \( p_R \) and excluding arguments involving only a constant in a slight abuse of notation. Comparing these scaling functions against those in Eq. (8), we identify the critical exponents

\[ \eta = \gamma_D^* - \gamma_D^*, \quad \eta' = -\gamma_D^*, \quad z = 2 + \gamma_D^* - \gamma^*_D. \]  

(D10)

Similarly, we can identify \( \nu^{-1} = -\gamma_j^* + \gamma_D^* \) although the subtleties of a complex-valued exponent \( \nu \) at the NEFPs are discussed in detail in the main text. Finally, we note that, to the first order in \( \epsilon \), we can neglect \( \gamma_D^* \) in the expression for \( \nu \).

[34] E. G. Dalla Torre, E. Demler, T. Giamarchi, and E. Altman, Quantum critical states and phase transitions in the presence of non-equilibrium noise, Nat. Phys. 6, 806 (2010).
[64] A. D. Bruce and A. Aharony, Coupled order parameters, symmetry-breaking irrelevant scaling fields, and
[104] R. Mottn, F. Brennecke, K. Baumann, R. Landig, T. Donner, and T. Esslinger, Roton-Type Mode Soft-

