Circuit Complexity across a Topological Phase Transition

Fangli Liu,1,2 Rex Lundgren,1 Paraj Titum,1,2 James R. Garrison,1,2 and Alexey V. Gorshkov1,2

1Joint Quantum Institute, NIST/University of Maryland, College Park, Maryland 20742, USA
2Joint Center for Quantum Information and Computer Science, NIST/University of Maryland, College Park, MD 20742, USA

We use Nielsen’s approach to quantify the circuit complexity in the one-dimensional Kitaev model. In equilibrium, we find that the circuit complexity of ground states exhibits a divergent derivative at the critical point, signaling the presence of a topological phase transition. Out of equilibrium, we study the complexity dynamics after a sudden quench, and find that the steady-state complexity exhibits nonanalytical behavior when quenched across critical points. We generalize our results to the long-range interacting case, and demonstrate that the circuit complexity correctly predicts the critical point between regions with different semi-integer topological numbers. Our results establish a connection between circuit complexity and quantum phase transitions both in and out of equilibrium, and can be easily generalized to topological phase transitions in higher dimensions. Our study opens a new avenue to using circuit complexity as a novel quantity to understand many-body systems.

In computer science, the notion of computational complexity refers to the minimum number of elementary operations necessary to implement a given task. This concept readily extends to quantum information science, where quantum circuit complexity is defined as the minimum number of gates necessary to implement a desired unitary transformation. The corresponding circuit complexity of a quantum state characterizes how difficult it is to construct a unitary transformation \( U \) to evolve a reference state \( |R\rangle \) to the desired target state \( |T\rangle = U |R\rangle \) [1, 2]. Nielsen and collaborators used a geometric approach to tackle the problem of quantum complexity by considering optimal quantum circuits [3–5]. Suppose that the unitary transformation \( U(t) \) is generated by some time-dependent Hamiltonian \( H(t) \), with the requirement that \( U(t_f) = U \) (where \( t_f \) denotes the final time). One then can characterize the quantum state complexity by imposing a cost functional \( F(H(t)) \) on the control Hamiltonian \( H(t) \). By choosing a cost functional that defines a Riemannian geometry in the space of circuits, the problem of finding the optimal control Hamiltonian synthesizing \( U \) then corresponds to finding minimal geodesic paths in a Riemannian geometry (which also quantify the complexity of \( |R\rangle \)) [3–5].

Recently, Nielsen’s approach has been adopted in high-energy physics to quantify the complexity of quantum field theory states [6–15]. This is motivated, in part, by previous conjectures that relate the complexity of the boundary field theory to the bulk space-time geometry, i.e. the so-called “complexity equals volume” [16, 17] and “complexity equals action” [18, 19] proposals. Myers and collaborators used Nielsen’s geometric approach to explicitly calculate the complexity of the ground state of a free scalar field [6], and found surprising similarities to the results of holographic complexity. A contemporary study by Chapman et al., which used the Fubini-Study metric to quantify complexity [20], gave similar results. Several recent works have generalized these studies to other states, including coherent states [8], thermofield double states [7, 11], and free fermion fields [12–14]. These works motivate us to ask the following question: What is the relationship between circuit complexity and quantum phase transitions? This question is related to the long-standing fundamental problem of preparing quantum states across critical points [21–23].

In this Letter, we consider the circuit complexity of a topological quantum system. In particular, we use Nielsen’s approach to study the circuit complexity of the Kitaev chain, a prototypical model which exhibits topological phase transitions and hosts Majorana fermions for open boundary conditions [24–29]. Strikingly, we find that, for ground states, the complexity exhibits a diverging derivative at quantum critical points, indicating that circuit complexity is sensitive to topological phase transitions and can be used to detect them. Furthermore, we investigate the growth of complexity following a sudden quantum quench. We find that the complexity quickly saturates to a constant, which shows nonanalytical behavior as a function of the post-quench Hamiltonian parameters. This demonstrates that circuit complexity can detect topological phase transitions both in and out of equilibrium.

Finally, we extend our results to the Kitaev model with long-range pairing [30–33] and show that circuit complexity can also detect topological phase transitions between phases with semi-integer topological numbers. Our work establishes a connection between circuit complexity and topological phase transitions and paves the way toward using complexity as a novel quantity to understand phase transitions in many-body interacting systems.

The model.—The one-dimensional Kitaev model is described by the following Hamiltonian [24, 25]:

\[
\hat{H} = -\frac{J}{2} \sum_{j=1}^{L} (\hat{a}_j^\dagger \hat{a}_{j+1}^\dagger + \text{H.c.}) - \mu \sum_{j=1}^{L} \left( \hat{a}_j^\dagger \hat{a}_j - \frac{1}{2} \right) + \frac{\Delta}{2} \sum_{j=1}^{L} (\hat{a}_j^\dagger \hat{a}_{j+1}^\dagger + \text{H.c.}),
\]

(1)

where \( J \) is the hopping amplitude, \( \Delta \) is the superconducting pairing strength, \( \mu \) is the chemical potential, \( L \) is the total number of sites (assumed to be even), and \( \hat{a}_j^\dagger (\hat{a}_j) \) creates (annihilates) a fermion at site \( j \). We set \( J = 1 \) without loss of generality and assume antiperiodic boundary conditions throughout this work.
Upon Fourier transforming the creation and annihilation operators, \( \hat{a}_j = \frac{1}{\sqrt{L}} \sum_{k} e^{-ik_{jk}} \hat{a}_{kn} \), Eq. (1) can be written in k-space,

\[
\hat{H} = - \sum_{k_n} [\mu + \cos k_n] \left( \hat{a}_{k,n} \hat{a}_{-k,n} - \hat{a}_{-k,n} \hat{a}_{k,n} \right) + i \Delta \sin k_n \left( \hat{a}_{k,n}^\dagger \hat{a}_{-k,n}^\dagger - \hat{a}_{-k,n} \hat{a}_{k,n} \right),
\]

(2)

where \( k_n = \frac{2\pi n}{L} (n + 1/2) \) with \( n = 0, 1, \ldots, L/2 - 1 \). The above Hamiltonian can be diagonalized analytically via a Bogoliubov transformation, which yields the following quasiparticle spectrum:

\[
\varepsilon_{k_n} = \sqrt{(\mu + \cos k_n)^2 + \Delta^2 \sin^2 k_n},
\]

(3)

The ground state of Eq. (1) can be written as

\[
|\Psi_{gs}\rangle = \prod_{n=0}^{L/2-1} (\cos \theta_{k_n} - i \sin \theta_{k_n} \hat{a}_{k,n}^\dagger \hat{a}_{-k,n}^\dagger) |0\rangle,
\]

(4)

where \( \tan(2\theta_{k_n}) = \Delta \sin k_n / (\mu + \cos k_n) \). Equation (3) shows that the quasiparticle excitation spectrum is gapless when \( |\mu| = 1 \) for arbitrary \( \Delta \) or when \( |\mu| \leq 1 \) and \( \Delta = 0 \). Indeed, quantum phase transitions occur at these straight lines, which separate distinct topological regions [24] [see Fig. 1(a) for the phase diagram]. The nontrivial topological phase is characterized by a nonzero winding number [Fig. 1(a)] and the presence of massless Majorana edge modes in an open chain [24–29].

**Complexity for a pair of fermions.**—From Eq. (2), we can see that only pairs of fermions with momenta \( \pm k_n \) are coupled to each other. Hence, we first quantify the circuit complexity of two such fermions [12–14]. The total circuit complexity can then be obtained by summing all momentum contributions [6, 20].

Without loss of generality, we consider the reference ("\( R \)"), and target ("\( T \)"") states with the same momentum but different Bogoliubov angles: \( |\psi_{R,T}\rangle = (\cos \theta_{k_n}^R - i \sin \theta_{k_n}^R \hat{a}_{k,n}^\dagger \hat{a}_{-k,n}^\dagger) |0\rangle \). Expanding the target state in the basis of \(|\psi_{R}\rangle \) and \(|\psi_{R}\rangle_\perp \) (i.e., the state orthogonal to \(|\psi_{R}\rangle \)), we have \( |\psi_{T}\rangle = \cos(\Delta \theta_k) |\psi_{R}\rangle - i \sin(\Delta \theta_k) |\psi_{R}\rangle_\perp \), where \( \Delta \theta_k = \theta_{k_n}^R - \theta_{k_n}^T \). Now the goal is to find the minimal circuit depth needed to achieve the unitary transformation which connects the target and reference states:

\[
U_k = \begin{bmatrix}
\cos(\Delta \theta_k) & -i e^{-i \phi} \sin(\Delta \theta_k) \\
-i e^{i \phi} \sin(\Delta \theta_k) & \cos(\Delta \theta_k)
\end{bmatrix},
\]

(5)

where \( \phi \) is an arbitrary phase. Nielsen approached this as a Hamiltonian control problem, i.e. finding a time-dependent Hamiltonian \( \hat{H}_k(s) \) that synthesizes the trajectory in the space of unitaries [3, 4]:

\[
U_k(s) = \mathcal{P} \exp \left[ \int_0^s ds \hat{H}(s) \right], \quad \hat{H}(s) = \sum_I Y_k^I(s) O_I
\]

(6)

with boundary conditions \( U_k(s = 0) = 1 \), and \( U_k(s = 1) = U_k \). Here, \( \mathcal{P} \) is the path-ordering operator and \( O_I \) are the generators of \( U(2) \). Nielsen’s idea is then to define a cost (i.e. ‘length’) functional for the various possible paths to achieve the unitary transformation \( U_k \) [3, 4, 6, 12].

\[
D[U_k] = \int_0^1 ds \sum_I |Y_k^I(s)|^2,
\]

(7)

and to identify the optimal circuit or path by minimizing this functional. The cost (i.e. ‘length’) of the optimal path is called the complexity \( C \) of the target state, i.e.

\[
C[U_k] = \min \{ Y_k^I(s) \} D[U_k].
\]

(8)

While we have chosen the particular cost functional in Eq. (7), one obtains similar results if the cost functional has a square root over the integrand [6, 12–14].

Following the procedures given by Refs. [12–14], one can explicitly calculate the circuit complexity for synthesizing the unitary transformation, Eq. (5). For quadratic Hamiltonians, it is a simple expression that depends only on the difference between Bogoliubov angles (see Supplemental Material [34]),

\[
C(|\psi_R\rangle \rightarrow |\psi_T\rangle) = |\Delta \theta_k|^2.
\]

(9)

Note that the complexity \( C \) for two fermions is at most \( \pi^2/4 \), since \( |\Delta \theta_k| \in [0, \pi/2] \). It reaches its maximum value when the target state has zero overlap with the reference state, i.e., when \( |\psi_T\rangle = |\psi_R\rangle_\perp \) up to a phase.

**Complexity for ground states.**—Given the circuit complexity for a pair of fermions, we can readily obtain the complexity of many-body wavefunctions in the noninteracting model. We choose the reference and target states to be ground states of Eq. (1), but with different parameters. The total unitary transformation that connects the two states is:

\[
|\Psi_{gs}^T\rangle = \prod_{n=0}^{L/2-1} U_{k_n} |\Psi_{gs}^R\rangle,
\]

(10)

where \( U_{k_n} \), given by Eq. (5), connects a pair of fermionic states with momenta \( \pm k_n \). As modes with different absolute momentum are not coupled, it is straightforward to obtain the total circuit complexity by summing the contributions from each momentum sector [6, 12–14].

\[
C(|\Psi_{gs}^R\rangle \rightarrow |\Psi_{gs}^T\rangle) = \sum_{k_n} |\Delta \theta_{k_n}|^2,
\]

(11)

where \( \Delta \theta_{k_n} \) is difference of the Bogoliubov angles between the reference and target states for momentum sector \( k_n \). In the infinite-system-size limit, the summation over \( k_n \) can be replaced by an integral, and one can derive that \( C \propto L \). This “volume law” dependence is reminiscent of the “complexity equals volume” conjecture in holography [16, 17], albeit in a different setting.
Equation (11) tells us that the circuit complexity between the target state and the reference state has a geometric interpretation, as it is the squared Euclidean distance in a high dimensional space [35]. Moreover, the geodesic path (or optimal circuit) in unitary space turns out to be the straight line connecting the two points (i.e. $H_k(s)$ independent of $s$) [34]. The circuit complexity can be compared with the winding number [25], another geometrical quantity which characterizes the number of times the Bogoliubov angle winds through the Brillouin zone. While the winding number is defined for a single state, the circuit complexity is defined as a difference between two states. In the remainder of this paper, we demonstrate that the circuit complexity is likewise able to reveal topological phase transitions.

We begin by investigating ground-state properties. We choose a fixed ground state as the reference state and calculate the circuit complexities for target ground states with various chemical potentials, $\mu_T$. As can be seen in Fig. 1(b), the circuit complexity is nonzero when the target state has a parameter (e.g. chemical potential) different from the reference state, and increases as the difference between the two parameters is increased. More importantly, the complexity grows rapidly around the critical points ($\mu_T = \pm 1$), changing from a convex function to a concave function at the critical points. This is further illustrated in Fig. 1(c), where we plot the derivative of circuit complexity with respect to chemical potential. There is clear divergence at the critical points (see Supplemental Material [34] for an analytical derivation). This indicates that circuit complexity is a nonanalytical function at the critical points, and thus can signal the presence of a quantum phase transition. We emphasize that these features are robust signatures of phase transitions, which do not change if one chooses a different reference state. As demonstrated in Figs. 1(b) and (c), the rapid changing of complexity and diverging derivatives persist for all three reference states we choose (ground states with different $\mu_R$).

To understand the above results, we plot the Bogoliubov angle difference, $\Delta \theta_{k_n}$, versus the momentum, $k_n$, for various target states (with a fixed reference state with $\mu_R = 0$) in Fig. 1(d). When both ground states are in the same phase region, $\Delta \theta_{k_n}$ first increases with momentum, and finally decreases to 0 when $k_n$ approaches $\pi$. In contrast, when $\mu_T$ is beyond its critical value, $\Delta \theta_{k_n}$ increases monotonically with momentum, and takes the maximal value of $\pi/2$ when $k_n = \pi$. This is related to the winding number analysis, where the Bogoliubov angles of two different states end up at the same pole (on the Bloch sphere) upon winding half of the Brillouin zone if the states belong to the same phase [25]. As the circuit complexity is an integral of $|\Delta \theta_{k_n}|^2$, the distinct dependence of the Bogoliubov angle difference on $k_n$ results in the different concave-convex character of circuit complexity before and after the phase transition. This again leads to nonanalytical behavior (divergence of the first-order derivative) of the circuit complexity at critical points [Fig. 1(c)].

Figure 2(a) shows the derivative of circuit complexity with respect to $\mu_T$ for a wide range of $\Delta_T$. As one can see, the derivatives show clear singular behavior at the critical points for all $\Delta_T$ [34], and map out the phase boundaries in the vertical direction, in perfect agreement with the phase diagram presented in Fig. 1(a). In Fig. 2(b), we plot the derivative of circuit complexity with respect to $\Delta_T$ [Figs. 2(a) and (b)] maps out the phase
boundaries of the Kitaev chain [Fig. 1(a)].

Complexity dynamics after a sudden quench.—We have shown that circuit complexity can detect topological phase transitions in ground states. Motivated by several recent studies using quench dynamics to detect topological phase transitions [36–40], as well as studies on dynamical phase transitions [41–45], we investigate whether the post-quench dynamics of circuit complexity can also detect topological phase transitions.

We take the initial state to be the ground state of a Hamiltonian \( \hat{H}_i \), and consider circuit complexity growth under evolution of a different Hamiltonian, \( \hat{H}_f \). The reference and target states are chosen as the initial state \( |\Psi_i\rangle \) and time-evolved state \( |\Psi(t)\rangle \) respectively. As the Hamiltonian can be diagonalized in momentum space, one can write the post-quench state as [46, 47]

\[
|\Psi(t)\rangle = \prod_{n=0}^{\xi-1} \left[ \cos(\Delta \theta_{k_n}) - i e^{2i \varepsilon_{k_n} t} \sin(\Delta \theta_{k_n}) \hat{A}_{k_n}^\dagger \hat{A}_{k_n} \right]|0\rangle,
\]

(12)

where \( \Delta \theta_{k_n} \) is the Bogoliubov angle difference at momentum \( k_n \) between eigenstates of the initial and post-quench Hamiltonians, and \( \varepsilon_{k_n} \) and \( \hat{A}_{k_n} \) are the energy levels and normal mode operators, respectively, for the post-quench Hamiltonian. Similar to the previous derivation of complexity for ground states, one can obtain the time-dependent circuit complexity,

\[
C(\langle \Psi_i \rangle \rightarrow |\Psi(t)\rangle) = \sum_{k_n} \phi_{k_n}^2(t)
\]

(13)

where \( \phi_{k_n}(t) = \arccos \sqrt{1 - \sin^2(2\Delta \theta_{k_n}) \sin^2(\varepsilon_{k_n} t)} \).

Figure 3(a) shows the circuit complexity growth for various post-quench Hamiltonians with fixed initial state. As one can see, the circuit complexity first increases linearly and then oscillates [9, 10, 15] before approaching a time-independent value, which can be derived from the time-averaged value of \( \phi_{k_n}(t) \). The steady-state value of circuit complexity increases with the chemical potential of the post-quench Hamiltonian, \( \mu_f \), until the phase transition occurs [Fig. 3(a)]. Interestingly, after the phase transition, the steady-state complexity saturates to a single value when \( \mu_i = 0 \), no matter how strong the quench is [Fig. 3(a)].

Figure 3(b) further illustrates the long-time steady-state values of circuit complexity for different initial states as a function of \( \mu_f \). One can clearly see that the steady-state complexity exhibits nonanalytical behavior at the critical point. This nonanalytical behavior arises because the time-averaged value of \( \phi_{k_n}(t) \) exhibits an upper bound after the phase transition (see Supplemental Material [34]), and such behavior is robust as it occurs for all initial states. Therefore, the steady-state value of circuit complexity after a quantum quench can reveal whether the initial and post-quench Hamiltonians are in different topological phases, just like for the ground state circuit complexity.

Generalization to long-range interactions.—While we have restricted ourselves to the one-dimensional Kitaev model with short-range interactions thus far, the results we found can be easily generalized to other topological systems. To give a specific example, we extend the original Hamiltonian, Eq. (1), to include long-range pairing [30–33],

\[
\hat{H}_{LR} = -\frac{J}{2} \sum_{j=1}^{L} (\hat{a}_{j+1}^\dagger \hat{a}_j + H.c.) - \mu \sum_{j=1}^{L} (\hat{a}_{j+1}^\dagger \hat{a}_j - \frac{1}{2})
\]

\[
+ \frac{\Delta}{2} \sum_{j=1}^{L} \sum_{\ell=1}^{L-1} d_\ell \left( \hat{a}_{j+1}^\dagger \hat{a}_j - \frac{1}{2} \right) + \mu \sum_{j=1}^{L} \left( \hat{a}_j^\dagger \hat{a}_j - \frac{1}{2} \right) + H.c.
\]

(14)

where \( d_\ell = \min(\ell, L - \ell) \). In contrast to the short-range model, the long-range model with \( \alpha < 1 \) hosts topological phases with semi-integer winding numbers and supports massive edge modes [30, 33].

Figure 4(a) shows the derivative of the circuit complexity (with respect to chemical potential) with \( \alpha = 0 \) (all-to-all pairing). Here the reference and target states are taken as...
the ground states of the long-range Hamiltonian. The nonanalytical behavior occurs only at $\mu_T = 1$, in contrast with Fig. 1(c). This agrees perfectly with the topological phase diagram, where a topological phase transition occurs only at $\mu = 1$ for $\alpha = 0$. This phase transition separates topological phases with winding number $W = \pm 1/2$ [33]. Figure 4(b) shows the long-time steady-state values of the circuit complexity after a sudden quench. Again, one observes nonanalytic behavior only at $\mu_T = 1$. This demonstrates that the circuit complexity of ground states and nonequilibrium steady states can both reveal topological phase transitions between phases with half-integer winding numbers.

Conclusions and outlook.—We have used Nielsen’s approach to quantify the circuit complexity of ground states and nonequilibrium steady states of the Kitaev chain with short- and long-range pairing, and found that, in both situations, circuit complexity can be used to detect topological phase transitions. Our results can be readily generalized to other higher-dimensional topological systems, such as topological insulators and topological superconductors [48, 49]. One interesting direction for future investigation is to use the geometric approach to quantify circuit complexity in real space [50–52] rather than momentum space, and study its relation to quantum phase transitions. It would also be interesting to study the circuit complexity of interacting many-body models. One such example is the XXZ spin-half chain, whose low-energy physics can be modeled by the Luttinger liquid Hamiltonian [53–55]. By restricting to certain classes of gates (i.e., by imposing penalties on the cost function) [3, 6], it might be possible to find improved methods to efficiently prepare the ground state of the XXZ model by calculating the geodesic path (optimal circuit) in gate space.

Note added: While in the final stages of preparing this manuscript, we became aware of Ref. [56], which used revivals of circuit complexity as a qualitative feature to detect phase transitions in the Su-Schrieffer-Heeger model. In contrast, here we have shown that the steady-state circuit complexity exhibits nonanalyticities precisely at critical points for the Kitaev chain. In addition, we have demonstrated that the complexity between ground states also shows nonanalyticity at topological phase transitions.

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[34] See Supplemental Material for detailed derivations of circuit complexity for a pair of fermions, analytical derivations of the nonanalyticity of circuit complexity at critical points, and numerical details for the nonanalyticity of circuit complexity after quantum quenches.

[35] In such a space, each state is represented by one point, with its coordinates labeled by the Bogoliubov angle of each momentum sector, i.e. \( \theta_{k_0}, \theta_{k_1}, \ldots, \theta_{k_{L/2-1}} \).


Supplemental Material

This Supplemental Material consists of three sections. In Sec. I, we analytically derive the circuit complexity for a pair of fermions [Eq. (9) of the main text]. In Sec. II, we provide detailed derivations of the nonanalyticity of circuit complexity at critical points, as shown numerically in Figs. 1(b) and (c) and Figs. 2(a) and (b) of the main text. In Sec. III, we provide numerical and analytical evidence of the nonanalyticity of steady-state circuit complexity after a quantum quench.

I. DERIVATION OF CIRCUIT COMPLEXITY FOR A PAIR OF FERMIONS

In this section, we present a detailed derivation of the circuit complexity for a pair of fermions, i.e. Eq. (9) in the main text. This expression has previously been obtained using different approaches in Refs. [12–14]. In order to be comprehensive, here we provide a detailed derivation following Ref. [13]. We note that Ref. [12] provides an alternative derivation using a group theory approach.

By taking the derivative with respect to \( s \) in Eq. (6) of the main text, we get the following expression:

\[
\sum_l Y_l^I(s) O_l = (\partial_s U(s)) U^{-1}(s),
\]

where \( U(s) \) is a unitary transformation which depends on \( s \), and we have omitted the label \( k \) for notational clarity.

The unitary \( U(s) \) can be parametrized in matrix form:

\[
U(s) = e^{i\beta} \begin{bmatrix} e^{-i\phi_1} \cos \omega & e^{-i\phi_2} \sin \omega \\ -e^{i\phi_2} \sin \omega & e^{i\phi_1} \cos \omega \end{bmatrix},
\]

where \( \beta, \phi_1, \phi_2, \omega \) explicitly depend on the parameter \( s \). The above matrix can be expressed in terms of the generators of \( U(2) \), which we choose as follows:

\[
O_0 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad O_1 = \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}, \quad O_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad O_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.
\]

Using the relation

\[
\text{tr}(O_a O_b) = -2\delta_{ab},
\]

one can extract the strength, \( Y_l^I(s) \), of generator \( O_l \) [cf. Eq. (6) in the main text] as follows:

\[
Y_l^I(s) = -\frac{1}{2} \text{tr} \left[ (\partial_s U(s)) U^{-1}(s) O_l \right].
\]

Our cost functional can then be expressed as

\[
D = \int_0^1 ds \sum_l |Y_l^I(s)|^2
= \int_0^1 ds \left[ \left( \frac{d\beta}{ds} \right)^2 + \left( \frac{d\omega}{ds} \right)^2 + \cos^2 \omega \left( \frac{d\phi_1}{ds} \right)^2 + \sin^2 \omega \left( \frac{d\phi_2}{ds} \right)^2 \right].
\]

Now, by exploiting the boundary condition at \( s = 0 \), i.e. \( U(s = 0) = I \), we get

\[
\begin{bmatrix} \beta(s = 0) \\ \phi_1(s = 0) \\ \phi_2(s = 0) \\ \omega(s = 0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \phi_2(0) \\ 0 \end{bmatrix}, \quad (S7)
\]

where \( \phi_2(0) \) is an arbitrary phase. Furthermore, we have the boundary condition at \( s = 1 \),

\[
U(s = 1) = \begin{bmatrix} \cos(\Delta \theta) & -ie^{-i\phi} \sin(\Delta \theta) \\ -i \sin(\Delta \theta) & e^{-i\phi} \cos(\Delta \theta) \end{bmatrix},
\]

which results in

\[
\begin{bmatrix} \beta(s = 1) \\ \phi_1(s = 1) \\ \phi_2(s = 1) \\ \omega(s = 1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \pi/2 \\ \Delta \theta \end{bmatrix}. \quad (S9)
\]

The integrand in Eq. (S6) is a sum of four non-negative terms. Setting \( \beta(s) = \phi_1(s) = 0 \) and \( \phi_2(s) = \pi/2 \) minimizes (i.e. sets to zero) three of the four terms without imposing any additional constraints on the minimization of the remaining \( d\omega/ds \) term. One can then easily check that the linear function \( w(s) = s\Delta \theta \) minimizes the remaining term and yields

\[
C = \int_0^1 ds |\Delta \theta|^2 = |\Delta \theta|^2. \quad (S10)
\]

II. ANALYTICAL DERIVATION OF DIVERGENT DERIVATIVES IN GROUND STATES

In this section, we provide a detailed analytical derivation to show that the first-order derivative indeed diverges at the critical points in the thermodynamic limit. We first show that the derivative diverges for a particular reference state \( (\mu_R = 0, \Delta_R = 1) \), and then we generalize our results to show that this divergent behavior does not depend on the particular choice of \( \mu_R \) for the reference state, in agreement with Figs. 1(b) and (c) in the main text. Finally, we provide analytical derivations showing that these divergences can indeed map out the phase boundary, as illustrated in Figs. 2(a) and (b) in the main text.

Let us start with a simple case, where the pairing strength of the reference state, in agreement with Eq. (5) of the main text, is treated as a variable. The Bogoliubov angle \( \Delta \theta_k \) for each momentum sector \( k \) can be expressed as

\[
\Delta \theta_k = \frac{1}{2} \arccos \frac{1 + \mu_T \cos k}{\sqrt{\mu_T^2 + 2\mu_T \cos k + 1}}, \quad (S11)
\]

and the circuit complexity is written in terms of \( \Delta \theta_k \):

\[
\frac{C}{L} = \frac{1}{2\pi} \int_0^\pi |\Delta \theta_k|^2 dk. \quad (S12)
\]
Note that we have replaced the discrete sum in the main text with an integral for the thermodynamic limit, and written \(\mathcal{C} [\langle \Psi_\mu \rangle \rightarrow \langle \Psi_\mu^* \rangle] \) as \(\mathcal{C}\) for brevity.

Now we substitute Eq. (S11) into Eq. (S12), and take the derivative with respect to \(\mu_T\) around the critical point \(\mu_T = 1\). We obtain

\[
\partial_{\mu_T} (C) / L = \frac{1}{16\pi} \int_0^\pi k \tan \left( \frac{k}{2} \right) dk. \tag{S13}
\]

The integral defined in the above equation diverges in a logarithmic fashion, with \(\lim_{k \to 0} \frac{1}{16\pi} \int_0^\pi k \tan (k/2) dk \approx -\frac{1}{8} \log \frac{1}{2}\). Intuitively, this behavior is due to the divergence of \(\tan (k/2)\) at \(k = \pi\). This also agrees with the abrupt change of \(\Delta \theta_{k_n}\) around \(k_n = \pi\) at the critical points, as shown in Fig. 1(d) in the main text. Therefore, we have proved the derivative diverges at this particular critical point. One can show that the derivative also diverges at \(\mu_T = -1\) by using a similar derivation.

Now let us fix \(\mu_R\) to be an arbitrary constant in the expression for \(\Delta \theta_k\), in order to show that the divergent behavior does not depend on the specific value of \(\mu_R\) chosen above. For simplicity, we take \(\Delta_R = \Delta_T = 1\), although one can also easily show that the results do not depend on the special choice of pairing strength. The expression for \(\Delta \theta_k\) can be written as

\[
\Delta \theta_k = \frac{1}{2} \arccos \frac{1 + (\mu_R + \mu_T) \cos k + \mu_T \mu_T}{\sqrt{\mu_R^2 + 2\mu_R \cos k + 1}}.
\]

The circuit complexity is given by substituting the above equation into Eq. (S12). By taking the first-order derivative around \(\mu_T = 1\), one gets

\[
\partial_{\mu_T} (C) / L = \frac{1}{8\pi} \int_0^\pi \arccos \frac{1 + \mu_R \cos (k/2)}{\sqrt{\mu_R^2 + 2\mu_R \cos k + 1}} \tan \left( \frac{k}{2} \right) dk. \tag{S15}
\]

Once again, because \(\tan (k/2)\) diverges while the integrand is nonzero at \(k = \pi\), the above expression indeed exhibits divergence at \(\mu_T = 1\) for arbitrary values of \(\mu_R\).

We now proceed to derive the more general divergent behavior exhibited in Fig. 2 of the main text. The fixed reference state is chosen with the parameters \(\mu_R = 0\), \(\Delta_R = -1\). Both \(\mu_T\) and \(\Delta_T\) are taken to be variables now. Again, \(\Delta \theta_k\) can be written as

\[
\Delta \theta_k = \frac{1}{2} \arccos \frac{(\cos k + \mu_T) \cos k - \Delta_T \sin^2 k}{\sqrt{(\mu_T + \cos k)^2 + \Delta_T^2 \sin^2 k}}. \tag{S16}
\]

By substituting the above expression into Eq. (S12) and taking the partial derivative of circuit complexity with respect to \(\mu_T\) around \(\mu_T = 1\), we get

\[
|\partial_{\mu_T} (C) / L| = \frac{1}{2\pi} \int_0^\pi \Delta_T \sin k \frac{\arccos \frac{\cos k + \cos^2 k - \Delta_T \sin^2 k}{(\mu_T + \cos k)^2 + \Delta_T^2 \sin^2 k}}{(1 + \cos k)^2 + \Delta_T^2 \sin^2 k} dk. \tag{S17}
\]

When \(k = \pi\), the expression \(\arccos \left[ \frac{\cos k + \cos^2 k - \Delta_T \sin^2 k}{(\mu_T + \cos k)^2 + \Delta_T^2 \sin^2 k} \right]\) is nonzero, while \(\frac{\Delta_T \sin k}{(\mu_T + \cos k)^2 + \Delta_T^2 \sin^2 k}\) diverges. Due to similar reasons as above, the overall integral does not converge at \(\mu_T = 1\). One can also see the divergent behavior at \(\mu_T = -1\) by following similar steps. This agrees with the numerical results in Fig. 2(a) of the main text.

To derive expressions corresponding to Fig. 2(b) in the main text, we take the derivative of circuit complexity with respect to \(\Delta_T\) around \(\Delta_T = 0\). We get

\[
|\partial_{\Delta_T} (C) / L| = \frac{1}{2\pi} \int_0^\pi \sin k \arccos \frac{\cos k (\cos k + \mu_T)}{|\mu_T + \cos k|} \frac{1}{|\mu_T + \cos k|} dk. \tag{S18}
\]

For \(|\mu_T| < 1\), there always exists a particular momentum \(k \in [0, \pi]\) for which the denominator is zero while the numerator is nonzero. This results in a divergent derivative for \(|\mu_T| < 1\). On the other hand, for \(|\mu_T| > 1\), there is no such divergent behavior, as the denominator always takes nonzero values. Therefore, the analytic derivations presented agree perfectly with the numerical results shown in Fig. 2(b) in the main text.

### III. NUMERICAL EVIDENCE FOR NONANALYTICITY OF QUENCH DYNAMICS

In this section, we provide detailed numerical explanations for the nonanalyticity of the long-time steady-state value of the circuit complexity at critical points, as observed in Fig. 3(b) of the main text.

As derived in the main text, the time-dependent circuit com-
the time-averaged value of the above expression, an analytical function for the time-averaged circuit complexity is given by

\[ C(\langle \Psi_i \rangle \rightarrow |\Psi(t)\rangle) = \sum_{k_n} \phi_{k_n}^2(t), \quad (S19) \]

where

\[ \phi_{k_n}(t) = \arccos \sqrt{1 - \sin^2(2\Delta \theta_{k_n}) \sin^2(\epsilon_{k_n} t)}. \quad (S20) \]

Then the long-time steady-state complexity is just given by the time-averaged value of the above expression,

\[ \overline{C(\langle \Psi_i \rangle \rightarrow |\Psi(t)\rangle)} = \sum_{k_n} \overline{\phi_{k_n}^2(t)}, \quad (S21) \]

where the overline denotes time averaging. Because \( \phi_{k_n}^2(t) \) is such a function of \( k_n \), it is unknown to us how to derive an analytical function for the time-averaged circuit complexity. Instead, we plot \( \phi_{k_n}(t) \) numerically, and show that the nonanalyticity indeed occurs at the phase transition.

From the expression of \( \phi_{k_n}(t) \), it is clear that its value oscillates with time, and it reaches its maximal value (envelope) for each momentum sector \( k_n \) when \( \sin(\epsilon_{k_n} t) = 1 \). In Fig. S1(a), we plot the maximum value of \( \phi_{k_n}(t) \) for different post-quench Hamiltonian parameters. As the figure clearly shows, when the chemical potential \( \mu_f \) of the post-quench Hamiltonian is below the critical value (\( \mu_f = 1 \)), \( \max[\phi_{k_n}(t)] \) is a smooth function of \( k_n \). However, when \( \mu_f \) is above the critical value, \( \max[\phi_{k_n}(t)] \) exhibits a kink at a certain momentum \( k_n \), with its maximal value reaching \( \pi/2 \). To understand this behavior, we can write down the expression for \( \max[\phi_{k_n}(t)] \) given the choice of parameters \( \mu_i = 0, \Delta_i = \Delta_f = 1 \):

\[ \max[\phi_{k_n}(t)] = \arccos \left| \frac{1 + \mu_f \cos k_n}{\sqrt{\mu_f^2 + 2\mu_f \cos k_n + 1}} \right|. \quad (S22) \]

From the above expression, it is clear that when \( \mu_f < 1 \), \( \max[\phi_{k_n}(t)] \) is always smaller than \( \pi/2 \); when \( \mu_f > 1 \), \( \max[\phi_{k_n}(t)] \) can obtain the maximal value of \( \pi/2 \) when \( 1 + \mu_f \cos k_n = 0 \). Because one needs to take the absolute value for the arguments of \( \arccos \), the quantity \( \max[\phi_{k_n}(t)] \) exhibits a kink when reaching \( \pi/2 \), in agreement with Fig. S1(a).

We plot the time-averaged value of \( \phi_{k_n}(t) \) in Fig. S1(b). Again, we see an upper bound of \( \phi_{k_n}(t) \) when quenching across the critical point. Similar to Fig. S1(a), \( \phi_{k_n}(t) \) reaches its maximal value when \( 1 + \mu_f \cos k_n = 0 \), i.e. when \( \sin(2\Delta \theta_{k_n}) = 1 \). For this special momentum sector, the expression for \( \phi_{k_n}(t) \) can be written as

\[ \phi_{k_n}(t) = \arcsin |\sin(\epsilon_{k_n} t)|. \quad (S23) \]

Clearly, the time-averaged value of the above expression is just \( \pi/4 \), in agreement with the numerical results shown in Fig. S1(b). Therefore, after the phase transition takes place, the maximal value of \( \phi_{k_n}(t) \) is bounded by \( \pi/4 \). (This feature is independent of the parameters of the pre-quench Hamiltonian.)

Having revealed this feature of \( \phi_{k_n}(t) \), the nonanalyticity can be understood as follows: as \( \mu_f \) increases but is still below the phase transition point, the integral of \( \phi_{k_n}^2(t) \) increases smoothly with \( \mu_f \). After reaching the phase transition, \( \phi_{k_n}^2(t) \) saturates the bound, and thus the integral’s (circuit complexity’s) dependence on \( \mu_f \) takes a different form. In particular, for the parameters shown in Fig. S1 (blue line in Fig. S1(b) in the main text), the integral (i.e., the circuit complexity) becomes a constant after the phase transition. This leads to a clear nonanalytical (kink) point at \( \mu_f = 1 \).